Proof of generalized Riemann hypothesis for Dedekind zetas and Dirichlet L-functions

Andrzej Mądrecki*

February 1, 2008

Abstract. A short proof of the generalized Riemann hypothesis (gRH in short) for zeta functions ζ_k of algebraic number fields k - based on the Hecke's proof of the functional equation for ζ_k and the method of the proof of the Riemann hypothesis derived in $[M_A]$ (algebraic proof of the Riemann hypothesis) is given. The generalized Riemann hypothesis for Dirichlet L-functions is an immediately consequence of (gRH) for ζ_k and suitable product formula which connects the Dedekind zetas with L-functions.

1 Introduction

Let k be an **algebraic number field**, (i.e. the main half of the set of **global fields**), i.e. a finite algebraic extension of the **rational number field** $\mathbb{Q}$. Let R_k be a ring of **algebraic integers** in k ,i.e. a finitely-generated ring extension - the integral closure - of the ring of **integers** $\mathbb{Z}$. Then, the **Dedekind zeta function** ζ_k for k is well locally defined (cf.e.g. [K, Chapter 7], [L,VIII.2] and [N, VII]) as the **Dirichlet series**

$$\zeta_k(s) := \sum_{0\neq I\in\mathcal{I}_k} \frac{1}{N(I)^s}\ ,\ Re(s)>1, \tag{1.1}$$

where by $\mathbb{C}$ we denote the field of all complex numbers and by $Re(s)$ and $Im(s)$ the **real** and **imaginary** part of a complex number s, respectively. We denote the **group of all fractional ideals** of the **Dedekind ring** R_k by $\mathcal{I}_k$ (cf.e.g. [N]) and finally $N(I)$ denotes the **absolute norm** of the ideal I, i.e. the number of elements in R_k/I.

We remark at once that we only use classical Dirichlet-Dedekind-Hecke theory, from the heroic period of German mathematics, to obtain an exciting result : a proof of the **generalized Riemann Hypothesis**(gRH_k in short) for algebraic number fields k. Hecke theory posseses such depth, that its classical tools are sufficient to obtain (gRH_k). For example, probably one of the most characteristic properties of the theory of classical number theory is that, one may embed a number field in the Cartesian product of its

*Institute of Mathematics, Wrocław University of Technology (WUT), 50-370 Wrocław, Poland

completions at the **archimedean points**, i.e. in a Euclidean space. In more recent years (more precisely since Chevalley introduced ideles in 1936, and Weil gave his adelic proof of the Riemann-Roch theorem soon afterwards), it has been found most convenient also to take the product over the **non-archimedean points**, with a suitable restriction on the components - the **adele ring** $\mathbb{A}_k$. However, we do not use the adele techique of **Tate's thesis** in this paper but stress **Hecke's theory** and we do not use the new achievements of algebraic number theory connected with adeles and ideles.

When, we work with Dedekind zetas, it is surprising that at once we obtain a very expanded apparatus of notions of the queen of mathematics - algebraic number theory.

The main property of ζ_k is the existence of the following **Hecke - Riemann analytic continuation functional equation** (HRace in short, cf.e.g. [L,XIII.3, Th.3])

$$\zeta_k^*(s) \ := \ \frac{\mid d(k) \mid^{s/2}}{2^{r_2 s}\pi^{ns/2}}\Gamma(\frac{s}{2})^{r_1}\Gamma(s)^{r_2}\zeta_k(s) \ = \ \frac{2^{r_1}h(k)R(k)}{w(k)s(s-1)} +$$

$$+ \sum_{0\neq I\in\mathcal{I}_k}\int_{||y||\geq 1} exp(-\pi d(k)^{-1/n}N(I)^{2/n}Tr(y))[\Pi(y)^{s/2} \ + \ \Pi(y)^{(1-s)/2}]\frac{dy}{y}$$

, where : $d(k)$ is the **discriminant** of a field k (cf.e.g.[N,II.2]),

$\Gamma(s) := \int_0^\infty e^{-x}x^{s-1}dx \ \ Re(s) > 0$ is the (classical) **gamma function**,

r_1 is the **number of real embeddings** of k into $\mathbb{C}$,

r_2 is **half of the number of complex embeddings** of k into $\mathbb{C}$, (The pair $r = [r_1, r_2]$ is called the **signature** of k).

$h(k)$ is the **class number** , $R(k)$ is the **regulator of** k (cf.[N, III.2]) and $w(k)$ is the number of **roots of unity** lying in k.

$S_\infty(k)$ denotes the set of **archimedean absolute values** of k, $n = n(k) = [k : \mathbb{Q}]$ is the **degree** of k over $\mathbb{Q}$,

$N_v(k) = N_v$ is the **local degree** of k, which is 1 if v is a real point of k and 2 if v is a complex valuation from the set $S_\infty(k)$.

Finally

$Tr_k(y) := \sum_{v\in S_\infty(k)} N_v y_v$ and $\prod(y) := \prod_{v\in S_\infty(k)} y_v^{N_v}$.

From the topological point of view the answer to the question : where are zeros and poles of ζ_k located - the algebraic number theory **characteristics (arithmetics invariant)** : $d(k), r_1, r_2, n(k), h(k), R(k), w(k), S_\infty(k)$ - which appears in (HRace) - (as we will show below) - are not so important, apart from the **topological invariants of** k , the signature $r(k)$, degree $n(k)$ and polynomial $s(s-1)$. For example, the invariants $h(k), R(k), w(k)$ and r_1 appear when we consider the **residue value** of ζ_k at the pole $s = 1$, but not when we consider the location of the **single pole** $\{1\} = I(\mathbb{C}) \cap R(\mathbb{C})$, where the algebraic varietes $I(\mathbb{C}) := \{s = u + iv \in \mathbb{C} : v(1-2u) = 0\}$ and $R(\mathbb{C}) := \{s = u + iv \in \mathbb{C} : u(u-1) - v^2 = 0\}$ **do not even depend on** k. Moreover, for the purposes of this paper it is only important that $h(k)$ is **finite**, but the value of $h(k)$ is not itself important. More exactly, we derive an essential generalization of (HRace), where

the n-dimensional **standard Gaussian function**

$$G_n(x) \ := \ e^{-\pi||x||_n^2} \ , \ x \in \mathbb{R}^n \tag{1.2}$$

(here $|| \ . \ ||_n$ is the **Euclidean norm** on $\mathbb{R}^n$ and obviously here, and all in the sequel, $\mathbb{R}$ stands for the field of real numbers), will be replaced by any smooth fixed point of $\mathcal{F}_n$.

The function G_n is a **fixed point** of the **Fourier transform** $\mathcal{F}_n$ on the **Schwartz space** $\mathcal{S}(\mathbb{R}^n)$ of smooth and rapidly decreasing functions. If we replace G_n by any other fixed point ω_+ of $\mathcal{F}_n$ from $\mathcal{S}(\mathbb{R})$, then we can extend the (HRace) to the **Fixed point Hecke Riemann analytic continuation equation** (Face in short) (cf. Section 2).

The idea of the generalization of (HRace) to (Face) is, in some small sense very similar to **Grothendieck's** magnificent idea of the generalization of the notion of set theory topology to category topologies (e.g. the well-known **etale cohomologies**) - to obtain the required results : to prove (gRH_k) in our case and to prove the **Riemann hypothesis** for **congruence Weil zetas**, respectively.

The following very important **rational function** (the **polar-zero part**) appears in HRace.

$$W_k(s) \ := \ \frac{\lambda_k}{s(s-1)}(:= \ \frac{2^{r_1}h(k)R(k)}{w(k)s(s-1)}) \ ; \ s \in \mathbb{C}. \tag{1.3}$$

Hence, in $W_k(s)$ is written a very important polynomial I of two variables, which does not depend on k! , with coefficients in $\mathbb{Z}$:

$$I(s) \ := \ Im(W_k(s)) \mid s(s-1) \mid^2 / \lambda_k \ = \ v(2u-1); \ s = u + iv, \ u, v \in \mathbb{R}. \tag{1.4}$$

The function $I(s)$ is mainly responsible for the form of the **generalized Riemann Hypothesis** for ζ_k ((gRH_k) in short), i.e. the following well-known implication (as in the case of the Riemann hypothesis , cf.$[M_A]$):

$$(gRH_k) \ \ If \ \zeta_k(s) = 0 \ and \ Im(s) \neq 0 \ , then \ Re(s) = 1/2.$$

According to (1.4) , the following **Trivial Riemann Hypothesis** ((TRH) in short) holds:

$$(TRH) \ If \ I(s) = 0 \ and \ Im(s) \neq 0 \ , then \ Re(s) = 1/2. \tag{1.5}$$

As in $[M_A]$ we pose the following **Algebraic conjecture for** ζ_k:

$$(TRH) \ implies \ (gRH_k).$$

More exactly, let us consider the **algebraic** $\mathbb{R}$**-variete** $I(\mathbb{C}) := \{s \in \mathbb{C} : I(s) = 0\}$ and the zero-dimensional **holomorphic manifold** $\zeta_k(\mathbb{C}) := \{s \in \mathbb{C} : \zeta_k(s) = 0\}$. Then the Riemann hypothesis (gRH_k) is a kind of relation between the cycles (of $\mathbb{R}^2$ and $\mathbb{C}$, respectively) : $I(\mathbb{C})$ (which does **not depend** on k) and $\zeta_k(\mathbb{C})$, i.e.

$$\zeta_k(\mathbb{C}) \ \subset \ I(\mathbb{C}).$$

In the sequel, the bi-affine-linear form $I(u,v)$ of two real variables, we call the **fundamental form** of the class $\{\zeta_k : k \ \ is\ an\ algebraic\ number\ field\}$.

Thus, **topological information** on the isolated points of the meromorphic function ζ_k is written - in fact - in the algebraic varieties $I(\mathbb{C})$ and $I(\mathbb{C}) \cap R(\mathbb{C})$, and therefore there exists some **unexpected** (and hence **deep**) relation between the **arithmetic** of $I \in \mathbb{Z}[u,v]$ over $\mathbb{R}$ and the **arithmetic** of ζ_k over $\mathbb{C}$. Moreover, the "serious" (gRH_k) could be reduced to the formal consequence of the "non-serious" (TRH) by calculating different kinds of integrals (with respect to different **Haar measures**), which leads to the **subsequence** functional equation : let $Gal(\mathbb{C}/\mathbb{R}) = \{id_{\mathbb{C}}, c\}$ be the **Galois group** of $\mathbb{C}$, i.e. $id_{\mathbb{C}}$ denotes the identity automorphism of $\mathbb{C}$ and c is the **complex conjugation** automorphism :

$$c(z) \ = \ c(u+iv) \ := \ u \ - \ iv, \tag{1.6}$$

which is an **idempotent map**, i.e. $c^2 = id_{\mathbb{C}}$. The following **generalized Riemann hypothesis functional equation** ($(gRhfe_k)$ in short) with a **rational term** I and the **action** of $Gal(\mathbb{C}/\mathbb{R})$ indicates some "hidden" Galois symmetry of ζ_k :

$$(gRhfe_k) \quad Im(\sum_{g\in Gal(\mathbb{C}/\mathbb{R})} (F_g\zeta_k)(g(s))) \ = \ \frac{\lambda_k(f_1(s)-f_2(s))I(s)}{\mid s(s-1)\mid^2} \ , \ Re(s) \in [0,1/2).$$

In opposite to the $(gRhfe_k)$, the (HRace) gives an "open symmetry" of ζ_k^* :

$$\zeta_k^*(s) \ = \ \zeta_k^*(1-s). \tag{1.7}$$

As in the case of the Riemann hypothesis, the functional equation $gRhfe_k$ - immediately implies the generalized Riemann hypothesis for the Dedekind zetas due to TRH. In comparison to $[M_A]$, we have significantly shorted the technicality of the proof of the theorem on existence of n-dimensional **RH-fixed points**. We consider the non-commutative field of **quaternions** $\mathbb{H}$, endowed with the Hilbert transform $\mathcal{H}_{\mathbb{H}}$ of a measure μ (see Sect.3)

$$(\mathcal{H}_{\mathbb{H}}\mu)(h) \ := \ \int_{\mathbb{H}} \frac{d\mu(x)}{\mid\mid h-x \mid\mid_4^4},$$

and the product ring (with zero divisors) $\mathbb{Q}_p \times \mathbb{Q}_q$ of different p-adic number fields endowed with the Hilbert transform $\mathcal{H}_{pq}$:

$$(\mathcal{H}_{pq}\mu)(a) \ := \ \int_{\mathbb{Q}_p\times\mathbb{Q}_q} \frac{d\mu(x)}{\Delta_{pq}(a-x)}.$$

Thus, using the techniques used in $[M_A]$ for the proof of the Riemann hypothesis, we show that our method initiated in that article works and can be significantly extended to the general case : this technique of RH-fixed points - leads to the proof of the generalized Riemann hypothesis for Dedekind zetas and Dirichlet L-functions.

The constructions in Section 3 are much more abstract in comparing to $[M_A]$ and much simpler. Moreover, these construct are interesting in themselves, since they (and in some sense return) to fundamental problems raused at the beginning of the 20th century.

The "heart" of the proof of RH from $[M_A]$ moving (practically without any changes) for gRH_k.

2 Fixed point Hecke-Riemann functional continuation equations

These two chapters achieve two goals simultaneously. We present here all the necessary preliminaries and notation. Next, we state the extension of (HRace) to (Face). Secondly, the main technical tool - and in fact - the "heart of the paper" , is Theorem 2 on the existence of multidimensional RH-fixed points. Moreover, we comnent on a surprising property of the construction mentioned: that it violates the Tertium non Datur in the case, when the **amplitude** A has a **support outside a set of Lebesgue measure zero**.

Let $n \in \mathbb{N}^* := \mathbb{N} - \{0\}$ be arbitrary (in all the sequel $\mathbb{N}^*$ denotes the set of all positive integers). In the sequel $n = n(k)$ will always be considered as the degree of a fixed algebraic number field k, i.e. $n = [k : \mathbb{Q}]$.

Exactly n different embeddings of k into the complex field $\mathbb{C}$ exists. Indeed, by **Abel's theorem** k can be written in the form $k = \mathbb{Q}(a)$ for a suitable **algebraic** a.

If $a_1, ..., a_n$ are all complex roots of the **minimal polynomial** for a over $\mathbb{Z}$, then the mappings $C_j, j = 1, ..., n$ (the **conjugates of** k) defined by

$$C_j(\sum_{k=0}^{n-1} A_k a^k) \ := \ \sum_{k=0}^{n-1} A_k a_j^k \tag{2.8}$$

(for $A_0, ..., A_{n-1} \in \mathbb{Q}$) are all isomorphisms of k **into** $\mathbb{C}$, and every such isomorphism has to be of this form.

The fields $C_j(k)$ are called the **fields conjugated** with k.

If $C_j(k) \subset \mathbb{R}$, then it is called a **real embedding** and otherwise $C_j(k)$ is called a **complex embedding**.

Note that if C_j is **complex**, then $c \circ C_j$ is again an embedding, complex of course, and so the number of complex embeddings is **even**. The number of such pairs of embeddings is usually denoted by $r_2(k) = r_2$, and the number of **real embeddings** by $r_1(k) = r_1$.

The pair $r = r(k) = [r_1, r_2]$ is called the **signature** of k (cf.e.g. [N, II.1])

We denote the **Lebesgue measure** on $\mathbb{R}^n$, and the **Lebesgue measure** of $\mathbb{C}^n$ by $d^n x$ and $d^n z$, respectively.

If $r = r(k) = [r_1, r_2]$ is the **signature of** k, then we define the **signature group** G_r of k as the product

$$G_r := \mathbb{R}_+^{r_1} \times (\mathbb{C}^*)^{r_2}, \tag{2.9}$$

of r_1 - exemplars of the multiplicative group $\mathbb{R}_+^*$ of **positive real numbers** and r_2-exemplars of the **multiplicative group of complex numbers** $\mathbb{C}^*$.

Obviously, G_r is a Locally Compact Abelian group (LCA in short). Hence, the **Haar measure** is well defined. Its **standardly normalized Haar measure** will be denoted by H_r. It is well-known that H_r is the product of the form :

$$dH_r(g) = \frac{d^{r_1}x}{\mid x \mid} \otimes \frac{d^{r_2}z}{\mid z \mid^2} = \otimes_{i=1}^{r_1} \frac{dx_i}{x_i} \otimes_{j=1}^{r_2} \frac{dz_j}{\mid z_j \mid^2}. \tag{2.10}$$

The signature group G_r is obviously the multiplicative subgroup of the **Euclidean ring**

$$E_r \ := \ \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \ \simeq \mathbb{R}^n, \tag{2.11}$$

with the componentwise multiplication. It is obviously a ring with divisors of zero. In particular, E_r has got the **Haar module** $\Delta_r = mod_r$ with the property

$$\Delta_r(g) \ = \ mod_r(g) \ = \ \prod_{i=1}^{r_1} \mid x_i \mid \prod_{j=1}^{r_2} \mid z_j \mid^2 , \ g = (x_1, ..., x_{r_1}, z_1, ..., z_{r_2}). \tag{2.12}$$

is well defined on E_r. Moreover

$$dH_r(g) \ = \ \frac{d^{r_1}x \otimes d^{r_2}z}{mod_r(g)}. \tag{2.13}$$

We denote the mod_r**-unit sphere** of G_r by G_r^0, i.e.

$$G_r^0 \ := \ \{g \in G_r : mod_r(g) = 1\}. \tag{2.14}$$

It is an elementary fact that we can write G_r as the product

$$G_r \ = \ \mathbb{R}_+^* \times G_r^0, \tag{2.15}$$

because any $g \in G_r$ can be written uniquely as

$$g \ = \ t^{1/n}c \tag{2.16}$$

with $t \in \mathbb{R}_+^*$ and $c \in G_r^0$. Here $c = \{c_v\}$ and $t^{1/n}c := (mod_r(c)^{1/n} \cdot (\frac{c_v}{mod_r c}))$.

We denote the **Haar measures** of G_r^0 by H_r^0. According to (2.15), the Haar measure H_r can be considered as the product of the **Lebesgue measure** dt/t on $\mathbb{R}_+^*$ and the appropriate **Haar measure** H_r^0 on G_r^0.

For a large class of Γ_r**-admissible** functions $f : G_r \longrightarrow \mathbb{C}$ the (n-dimensional) **Mellin transform** $M_n(f)$ or rather the **signature Gamma** $\Gamma_r(f)$ (associated with f) is well-defined as

$$\Gamma_r(f)(s) \ := \ \int_{G_r} mod_r^s(g) f(g) dH_r(g) \ =: \ M_n(f)(s) \ , \ Re(s) > 0. \tag{2.17}$$

Recall that $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ belongs to the **Schwartz space** $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing functions, if for each n-tuple of integers $\geq 0, k = (k_1, ..., k_n)$ and $l = (l_1, ..., l_n)$

$$p_{k,l}(f) \ := \ sup_{x \in \mathbb{R}} \mid x^k (D^l f)(x) \mid < +\infty,$$

where $x^k := x_1^{k_1}...x_n^{k_n}$ and $D^l := D_1^{l_1}...D_n^{l_n}$, is a partial differential operator.

It is easy to check (cf.e.g. $[M_A]$, Sect.2, Lemma1]) that the following holds for $f \in \mathcal{S}(\mathbb{R}^n)$:

$$\Gamma_n(f)(s) \;\in\; \mathbb{C} \;\; if \;\; Re(s) > 0, \tag{2.18}$$

since $\mathcal{S}(\mathbb{R}) \otimes ... \otimes \mathcal{S}(\mathbb{R})$ (n-times), is dense in $\mathcal{S}(\mathbb{R}^n)$.

We denote the (n-dimensional) **Fourier transform** of f by $\mathcal{F}_n f$ (for $\mathcal{F}$-admissible functions):

$$\mathcal{F}_n(f)(x) \;:=\; \int_{\mathbb{R}^n} e^{2\pi ixy} f(y) d^n y \;=:\; \hat{f}(x) \; ; \; x \in \mathbb{R}^n, \tag{2.19}$$

where $xy := \sum_{k=1}^n x_i y_i$ is the standard euclidean scalar product of n-vectors $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$. In this paper, it is also very convenient to use the (1-dimensional) **plus-Sin transform** defined as

$$S_+(f)(x) \;:=\; \int_0^{+\infty} sin(xy) f(y) dy \;=:\; \hat{f}_+(x) \;:\; x \in \mathbb{R}_+. \tag{2.20}$$

For another large class of θ**-admissible** functions $f : G_r \longrightarrow \mathbb{C}$ (n-dimensional or signatural), the**Jacobi theta function** $\theta_r(f)$ **associated with** f is defined as the series

$$\theta_r(f)(x) \;:=\; \sum_{k \in (\mathbb{N}^*)^n} f(k \cdot x) \;=\; \int_{(\mathbb{N}^*)^n} f(k \cdot x) dc(x) \;, x \in \mathbb{R}_+^n, \tag{2.21}$$

where $k{\cdot}x$ denotes componentwise multiplication in E_r and dc is the **calculating measure** on $(\mathbb{N}^*)^n$, i.e. the unique Haar measure on $\mathbb{Z}^n$ normalized by the condition : $c(\{0\}) = 1$.

Beside the field $\mathbb{C}$, we will also use the non-commutative field of quaternions $\mathbb{H}$. It is well-known (cf.e.g. [W]) that the formula

$$\Delta_{\mathbb{H}}(h) \;:= || \; h \; ||_4^4 \;, h \in \mathbb{H}, \tag{2.22}$$

defines the **Haar module** of $\mathbb{H}$.

For a class of some $\mathcal{H}$-admissible measures defined on a compact subset C of $\mathbb{H}$, we define the (compact) $\mathbb{H}$-**Hilbert transform** $\mathcal{H}_{\mathbb{H}}$ by the formula

$$(\mathcal{H}_{\mathbb{H}}\mu)(h) \;:=\; \int_C \frac{d\mu(x)}{\Delta_{\mathbb{H}}(h-x)} \;,\; h \in \mathbb{H}. \tag{2.23}$$

Finally, we use the product ring $\mathbb{Q}_p \times \mathbb{Q}_q$ with zero divisors of different p-adic number fields. It is well-known that the formula

$$\Delta_{pq}(x_p, x_q) \;:= |\; x_p \;|_p | \; x_q \;|_q \;, (x_p, x_q) \in \mathbb{Q}_{pq},$$

defines the **Haar module** of $\mathbb{Q}_{pq}$ and the formula

$$(\mathcal{H}_{pq}\mu)(a) \;:=\; \int_{\mathbb{Q}_p \times \mathbb{Q}_q} \frac{d\mu(x)}{\Delta_{pq}(a \;-\; x)} \;,\; a \in \mathbb{Q}_{pq},$$

defines pq-Hilbert transform.

Finally, we note that the Schwartz spaces $\mathcal{S}(\mathbb{R}^n)$ are **admissible** for all the integral transforms defined above : $\Gamma_r, \mathcal{F}_n, \theta_r$ and $\mathcal{H}$ (for absolutely continuous measures μ w.r.t. Lebesgue measure d^4h and the Haar measure dH_{pq} of $\mathbb{Q}_{pq}$, considered as densities of signed measures).

One of the main tools when we work with zetas is the **Poisson Summation Formula** (PSF in short, cf.e.g. [N], [L, XIII.2]) , which shows that $\mathcal{F}_n$ is a $l^1(\mathbb{Z})$**-quasi-isometry** on $\mathcal{S}(\mathbb{R}^n)$ and using our notation can be written as :

$$(PSF) \quad \int_{\mathbb{Z}^n} \hat{f}(x)dc(x) \ = \ \int_{\mathbb{Z}^n} f(x)dc(x),$$

if $f \in \mathcal{S}(\mathbb{R}^n)$.

A complex function ω_+ on $\mathbb{R}^n$ $(n = r_1 + 2r_2)$ is called a **fixed point of** $\mathcal{F}_n$, if it is an **eigenvector** of $\mathcal{F}_n$ with the corresponding **eigenvalue** equal to 1, i.e.

$$\mathcal{F}_n(\omega_+) \ = \ \hat{\omega_+} \ = \ \omega_+. \tag{2.24}$$

Analogously a complex valued function ω_- on $\mathbb{R}^n$ is called the **-fixed point** of $\mathcal{F}_n$ if it is an **eigenvector** of $\mathcal{F}_n$ corresponding to the **eigenvalue** -1 of $\mathcal{F}_n$:

$$\mathcal{F}_n(\omega_-) = \hat{\omega_-} = -\omega_-.$$

We use the common name for +fixed points and $-$fixed point - the $\pm$**fixed points** $\omega_\pm$.

Let $\omega = \omega_\pm$ be a $\pm$**fixed point** of $\mathcal{F}_n$ from $\mathcal{S}(\mathbb{R}^n)$ and let $M = [m_{ij}]_{n\times n}$ be a **matrix** of real numbers.

Let us consider the function

$$\omega_M(x) \ := \ \omega_\pm(Mx^t) \ \ ; \ \ x^t \in \mathbb{R}^n,$$

and the theta associated with it

$$\theta_n(\omega_M)(x) \ := \ \sum_{m\in\mathbb{Z}^n} \omega_M(mx) \ , \ x \in \mathbb{R}^n. \tag{2.25}$$

Lemma 1 *(***Hecke's theta formula***)*

For each non-singular matrix M the following relation holds

$$(HTF) \quad \theta_n(\omega^\pm_M)(x) \ \ = \ \ \pm\theta_n(\omega^\pm_{{}^tM^{-1}})(x)/ \mid det(M) \mid .$$

Proof. Let $M = [m_{ij}]_{n\times n}$ and $\omega_M(x) := \omega(Mx^t), x \in \mathbb{R}^n$. If M is a non-singular real matrix, then $det(M) \neq 0$. Thus the function ω_M is also in $\mathcal{S}(\mathbb{R}^n)$ and using the change of variables formula for multiple integrals, we immediately find that its Fourier transform is given by

$$\hat{\omega^\pm_M}(x) \ = \ \pm\frac{\omega({}^tM^{-1}x^t)}{\mid det(M) \mid},$$

where ${}^tM^{-1}$ is the transpose of the inverse of M.

This is clear, since when we make the change of variables $z = Mx^t$, we have $dz =| det(M) | dx$, and $< M^{-1}z^t, y >=< z,^t M^{-1}y^t >$.

The first important step in the proof of (gRH_k) is the **generalization of the Hecke-Riemann analytic continuation eqation** ((HRace) in short), given below. Therefore we need some additional notation.

Let us again consider the signature **Euclidean ring**

$$E_r \ = \ \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \ \simeq \mathbb{R}^n,$$

and the **conjugation** map $C : k \longrightarrow E_r$ defined as

$$C(\xi) \ := \ (C_1(\xi), ..., C_n(\xi)) \ ; \xi \in k. \tag{2.26}$$

Let us observe that each conjugate C_v determines the **absolute value** (place) v of k by the formula :

$$v(\xi) \ := | \ C_v(\xi) \ | \ ; \ \xi \in k. \tag{2.27}$$

The **completion** of (k, v) is denoted by k_v. Since v is **archimedean**, k_v is equal to $\mathbb{R}$ or $\mathbb{C}$.

In the case : $k_v \simeq \mathbb{C}$ the completion is determined up to **complex conjugation** c, according to the well-known elementary fact that if $\sigma \in Gal(\mathbb{C}/\mathbb{R})$, then

$$v(\sigma(z)) \ = \ v(z) \ , \ z \in \mathbb{C} \tag{2.28}$$

(cf.e.g. [L, II.1] and [N, L.3.1]).

We denote the set of all **non-equivalent archimedean places** of k by $S_\infty(k)$. According to (2.28), it is obvious that

$$| \ S_\infty(k) \ | = \ r_1 \ + \ r_2.$$

Let us consider the map $| \ C \ |: k \longrightarrow | \ E_r \ | := \ \mathbb{R}^{r_1+r_2}$ defined as

$$| \ C \ | \ (\xi) \ := \ (| \ C_v(\xi) \ |: v \in S_\infty(k)). \tag{2.29}$$

Recall that $G_r = (\mathbb{R}^*_+)^{r_1} \times (\mathbb{C}^*)^{r_2}$. So, if we denote : $| \ G_r \ | := (\mathbb{R}^*_+)^{r_1+r_2}$, then we have the decomposition

$$G_r \ \simeq | \ G_r \ | \times \mathbb{T}^{r_2}, \tag{2.30}$$

where $\mathbb{T} := \{z \in \mathbb{C} :| \ z \ |= 1\}$ is the 1-dimensional torus.

The kernel of $| \ C \ |$, i.e. $\mu(k) := ker(| \ C \ |)$ is the **group of the roots of unity** in k. Let

$$w(k) \ = \ \#\mu(k) \ \ = | \ \mu(k) \ |, \tag{2.31}$$

be the **number of roots of unity in** k.

Let $U(k)$ be the **group of units of** k ($S_\infty(k) -$ **units**)), i.e.

$$U(k) \ = \ R_k^*.$$

Let $V(k) \ := \mid C \mid (U(k))$ be the **image** of $U(k)$ under the mapping $\mid C \mid$. Its image $V(k)$ is contained in the subgroup $\mid G_r^0 \mid$ consisting of all $g \in \mid G_r \mid$ such that $mod_r(g) = 1$, and is a **discrete subgroup**. Furthermore, $\mid G_r^0 \mid /V(k)$ is **compact** (cf. [L,p.256]). Also, we can write G_r as the product

$$G_r \ = \ \mathbb{R}_+^* \times \mid G_r^0 \mid \times \mathbb{T}^{r_2}.$$

Finally, let $E(k)$ be the **fundamental domain for** $V^2(k)$ in $\mid G_r^0 \mid$ (cf. [L]). We obtain the following **disjoint decomposition**

$$\mid G_r^0 \mid = \ \cup_{\eta \in V} \eta^2 E(k). \tag{2.32}$$

Let A be an arbitrary **integral (fractional) ideal** of k. Then , it is well-known that A has an **integral basis** over $\mathbb{Z}$ (cf.[N, Th.2.4]). Thus, each $\xi \in A$ can be written as

$$\xi \ = \ x_1\alpha_1 \ + \ ... \ + \ x_n\alpha_n \ \ , \ \ x_i \in \mathbb{Z}. \tag{2.33}$$

For $v \in S_\infty(k)$ we let C_v be the embedding (conjugate) of k in k_v, identified with $\mathbb{R}$ or $\mathbb{C}$ (in the case of $\mathbb{C}$, we fix one identification, which otherwise is determined only up to conjugacy). We will write

$$\xi_v \ = \ C_v(\xi) \ = \ \sum_{j=1}^{n} x_j C_v(\alpha_j)$$

and

$$C(A) \ := \ [N(A)^{-1/n} C_i(\alpha_j)] \ , i,j = 1,...,n.$$

Hence, $N(A)^{-1/n}[\xi_1,...,\xi_n] = C(A)[x_1,...,x_n]^t$ and we also use this same notation when we constrict x_i to the set of real numbers.

Let $\mathcal{R}$ be an class of ideals of the ordinary ideal class group $H(k) := \mathcal{I}_k/P_k$. Let A be an ideal in $\mathcal{R}^{-1}$. The map

$$B \longrightarrow AB \ = \ (\xi) \tag{2.34}$$

eatablishes a bijection between the set of ideals in $\mathcal{R}$ and equivalence classes of non-zero elements of $A : \ A/ \sim_u$, where two field elements are called **equivalent** $\sim_u$, if they differ by a **unit**.

Let $R(A)$ be a set of **representatives** for the non-zero equivalence classes.

Finally, we introduce two thetas - small and capital : the **small Jacobi theta of** k (associated with ω)

$$\theta_k(\omega)(g) \ \ := \ \ \sum_{0 \neq I \in \mathcal{I}_k} \sum_{\xi \in R(I)} \sum_{u \in U(k)} \omega(u\xi g) \ = \tag{2.35}$$

$$= \sum_{0 \neq I \in \mathcal{I}_k} \sum_{x \in \mathbb{Z}^n} \theta_n(\omega_{C(I)})(g) \; ; \; g \in G_r,$$

and the **radial Jacobi theta** of k

$$\Theta_k(\omega)(t) \; := \; \frac{\int_{E(k)} \theta_k(\omega)(ct^{1/n}) dH_r^0(c)}{w(k)} \; , \; t \in \mathbb{R}_+^*.$$

Theorem 1 *(***Fixed point HRace = Face***) The following functional equation holds for each $\pm$fixed point $\omega_\pm$ of $\mathcal{F}_n$ from $\mathcal{S}(\mathbb{R}^n)$, with the property that $\Gamma_r(\omega_\pm)$* **does not vanishes***, and for each s with $Re(s) > 0$*

$$(Face) \;\; (\Gamma_r(\omega_\pm)\zeta_k)(s) \; = \; \frac{\lambda_k \neq 0}{s(s-1)} + \tag{2.36}$$

$$+ \int_1^\infty \int_{E(k)} \theta_k(\omega_\pm)(ct^{1/n})(t^s \; \pm \; t^{1-s}) dH_r^0(c) \frac{dt}{t} = \int_1^\infty \Theta_k(\omega_\pm(t)(t^{s-1} + t^{-s}))dt.$$

Proof. (A topological simplification of Lang's version of Hecke's proof of (HRace)). Let $\mathcal{R}$ be an **ideal class** of the ordinary **ideal class group** $H(k) := \mathcal{I}_k / P_k$, where P_k is the subgroup of principal fractional ideals. It is convenient to deal at initially with the zeta function associated with an ideal class $\mathcal{R}$. We define

$$\zeta_k(s, \mathcal{R}) \; := \; \sum_{B \in \mathcal{R}} \frac{1}{N(B)^s} \tag{2.37}$$

for $Re(s) > 1$. Let A be an ideal in $\mathcal{R}^{-1}$. Then the map

$$B \longrightarrow AB \; = \; (\xi) \tag{2.38}$$

establishes a bijection between the set of ideals in $\mathcal{R}$ and **equivalence classes of non-zero elements of** A (where two field elements are called equivalent, if they differ by a unit from $U(k)$). Let $R(A)$ be a set of **representatives** for the non-zero equivalence classes. Then

$$N(A)^{-s} \zeta_k(s, \mathcal{R}) \; = \; \sum_{\xi \in R(A)} mod_r(\xi N(A)^{-1/n})^{-s}. \tag{2.39}$$

We recall that the signature gamma is represented by the following integral (cf.(2.17))

$$\Gamma_r(\omega_\pm)(s) \; = \; \int_{G_r} \omega_\pm(g) mod_r(g)^s dH_r(g),$$

for $Re(s) > 0$, since

$$mod_r(\xi) \; = \; \prod_{v \in S_\infty(k)} \mid \xi_v \mid^{N_v},$$

where $N_v = [k_v : \mathbb{R}]$ are local degrees.

It will also be useful to note that if f is a function such that $f(g)/mod_r(g)$ is absolutely integrable on G_r, then

$$\int_{G_r} f(g)\frac{dH_r(g)}{mod_r(g)} \ = \ \int_{G_r} f(Mg)\frac{dH_r(g)}{mod_r(g)},$$

for any n-dimensional matrix $M = [m_{ij}]$ with **real** m_{ij}.

In other words, $dH_r(g)/mod_r(g)$ is an **invariant measure** of the **dynamical system** $(G_r, T_M(y) := My)$ or, in other words, H_r/Δ_r is a **Haar measure** on the group G_r.

Note that the signatural gamma function is expressed as such an integral.

Therefore, substituting g by $N(A)^{1/n}\xi g$ in (2.17), we obtain

$$\Gamma_r(\omega_\pm)(s)\frac{N(A)^s}{mod_r(\xi)^s} \ = \ \int_{G_r} \omega_\pm(\xi N(A)^{-1/n}g) mod_r(g)^s dH_r(g) \tag{2.40}$$

For $Re(s) \geq 1+\delta$, the sum over inequivalent $\xi \neq 0$ is absolutely and uniformly convergent. Since for $Re(s) > 1$,

$$N(A)^{-s}\zeta_k(s,\mathcal{R}) \ = \ \sum_{\xi\in R(A)} mod_r(\xi N(A)^{-1/n})^{-s},$$

it follows that

$$\Gamma_r(\omega)(s)\zeta_k(s,\mathcal{R}) = \int_{G_r} \sum_{\xi\in R(A)} \omega_{D(\xi)N(A)^{-1/n}}(g) mod_r(g)^s dH_r(g), \tag{2.41}$$

where $D(\xi) := [\delta_{iv}C_v(\xi)]$ denotes a diagonal matrix of conjugations. But according to (2.30), we can write

$$g \ = \ t^{1/n}c \ , \ t>0, c\in G_r^0.$$

Therefore,

$$\Gamma_r(\omega_\pm)(s)\zeta_k(s,\mathcal{R}) = \int_0^\infty \int_{G_r^0} \sum_{\xi\in R(A)} mod_r(t^{1/n}c)^s \omega_\pm(N(A)^{-1/n}(\xi_1 t^{1/n}c_1, ..., \xi_n t^{1/n}c_n)) t^s dH_r^0(c)\frac{dt}{t}, \tag{2.42}$$

where $dH_r^0(c)$ is the appropriate measure on G_r^0 and $c = (c_v)$ is a variable in G_r^0.

According to the decomposition (2.30) and since the kernel of $\mid C \mid$ is the group $\mu(k)$, we obtain from the above equation

$$\Gamma_r(\omega_\pm)(s)\zeta_k(s,\mathcal{R}) = \int_0^\infty \int_{E(k)} \frac{t^s}{w(k)} \sum_{u\in U(k)} \sum_{\xi\in R(A)} \omega_\pm(N(A)^{-1/n}(C_1(\xi u)t^{1/n}e_1, ..., C_n(\xi u)t^{1/n}e_n)) dH_{G_0}(e)dt/t = \tag{2.43}$$

$$= \frac{1}{w(k)}\int_0^\infty \int_{E(k)} t^s \sum_{u\in U(k)} \sum_{x\in X(A)} \omega_\pm((N(A)^{-1/n}(\sum_{j=1}^n x_j C_1(\alpha_j u)t^{1/n}e_1, ..., \sum_{j=1}^n x_j C_n(\alpha_j u)t^{1/n}e_n)) dH_{G_0}(e)dt/t,$$

where the second sum is over a **subset** $X(A)$ of $\mathbb{Z}^n - \{0\}$. But, according to the definition of $R(A)$, operating units we obtain that if u runs $U(k)$ and ξ runs $R(A)$ then $x = [x_1, ..., x_n] \in \mathbb{Z}^n - \{0\}$ from

$$u\xi \; = \sum_{x \in \mathbb{Z}^n - \{0\}} x_j \alpha_j$$

spars all $\mathbb{Z}^n - \{0\}$. Therefore, the "fourth integral" from (2.43) we can rewrite in the form

$$= \int_0^\infty \int_{E(k)} \frac{t^s}{w(k)} \sum_{x=(x_1,...,x_n)\neq 0} \omega_\pm((N(A)^{-1}t)^{1/n} e_1 \sum_{j=1}^n x_j C_1(\alpha_j, ..., (N(A)^{-1}t)^{1/n} e_n \sum_{j=1}^n C_n(\alpha_j)))dH_r^0(e)dt/t = \tag{2.44}$$

$$= \int_0^\infty \int_{E(k)} (\frac{t^s}{w(k)}) \sum_{0 \neq x \in \mathbb{Z}^n} \omega((N(A)^{-1}t)^{1/n} eC(A)x^t)dH_r^0(e)dt/t \; =$$

$$= \; \int_0^\infty \int_{E(k)} \frac{\theta_n(\omega_{C(A)})(t^{1/n}e) - 1}{w(k)} dH_r^0(e)dt/t.$$

We split the integral from 0 to ∞ into two integrals, from 0 to 1 and from 1 to ∞. We thus find

$$\Gamma_r(\omega_\pm)(s)\zeta_k(s, \mathcal{R}) \; = \; \frac{1}{w(k)} \int_0^1 t^s \int_{E(k)} \theta_n(\omega_{C(A)}(t^{1/n}e))t^s dH_r^0(e)dt/t - \tag{2.45}$$

$$- \; \frac{H_{G_r^0}(E(k))}{w(k)s} \; + \; \int_1^\infty \int_{E(k)} \frac{t^s}{w(k)} [\theta_n(\omega_{C(A)}(t^{1/n}e) - 1]dH_r^0(e)dt/t.$$

We return to the basis $\{\alpha_j : j = 1, ..., n\}$ of the integral ideal A over $\mathbb{Z}$. We define

$$\alpha^* \; := \; \{\alpha_j^* : j = 1, ..., n\}$$

to be the **dual basis** with respect to the trace (cf. [L, XII.3]). Then α^* is a basis for the fractional ideal

$$A^* \; := \; (D_{k/\mathbb{Q}}A)^{-1},$$

where $D_{k/\mathbb{Q}}$ is the **different** of k over $\mathbb{Q}$ (cf. [L, III.1], [N, IV.2] and the remark below).

We now use Heckes's theta functional equation (HTE). It can be seen that

$$\theta_n(\omega_{C(A)}^\pm)(t^{1/n}c) \; = \; \pm\frac{1}{t}\theta(\omega_{C(A^*)}^\pm)(t^{-1/n}c^{-1}), \tag{2.46}$$

because $mod_r(c) = 1$, i.e. c is in G_r^0 ! We transform the first integral from 0 to 1, using a simple change of variables, letting $t = 1/\tau, dt = -d\tau/\tau^2$. Note that the measure $dH_r^0(c)$ is invariant under the transformation $c \longrightarrow c^{-1}$ (think of an isomorphism with the additive Euclidean measure, invariant under taking negatives).

We therefore find that

$$\Gamma_r(\omega_\pm)(s)\zeta_k(s, \mathcal{R}) \; = \; \frac{2H_r^0(E(k) \times \mathbb{T}^{r_2})}{w(k)s(s-1)} + \tag{2.47}$$

$$+\frac{1}{w(k)}\int_1^\infty\int_{E(k)}(\theta_n(\omega_{C(A)})(t^{1/n}c)t^s \ \pm\ \theta_n(\omega_{C(A^*)})(t^{1/n}c)t^{1-s})dH_r^0(c)\frac{dt}{t}.$$

(Let us remark that in the second edition of [L] in Section XII.3 , on page 257 there is a typegraphical error).

The expression in (2.47) is **invariant** under the transformations
$A \longrightarrow A^*$ and $s \longrightarrow 1-s$ (in the plus case).

Thus, we have obtained full calculations on the zeta function of an ideal class $\mathcal{R}$. Taking the sum over the ideal classes $\mathcal{R}$ from $H(k)$ we immediately yield information on the zeta function itself, as follows : we can construct for A^* in a similar way ,and hence we finally obtain

$$2(\Gamma_r(\omega_\pm)\zeta_k)(s) = \Gamma_r(\omega_\pm)(s)(\sum_{\mathcal{R}\in H(k)} 2\zeta_k(s,\mathcal{R})) = \tag{2.48}$$

$$= \frac{\lambda_k}{s(s-1)} + \frac{1}{w(k)}\int_1^\infty\int_{E(k)}(t^s \pm t^{1-s})(\sum_{\mathcal{R}\in H(k)}\sum_{A\in\mathcal{R}^{-1}}\theta_n(\omega_{C(A)}(t^{1/n}c))d^*c\frac{dt}{t} =$$

$$= \frac{\lambda_k}{s(s-1)} + \frac{1}{w(k)}\int_1^\infty\int_{E(k)}(t^s \pm t^{1-s})\theta_k(\omega_\pm)(t^{1/n}c)dH_r^0(c)\frac{dt}{t}.$$

Remark 1 *As we mentioned above, in algebraic number theory we have to deal with a very expanded notional aparatus. We recall some ideas, explored in this paper.*

Let k be an arbitrary algebraic number field. Then we denote the **trace** *of k over $\mathbb{Q}$ by tr_k.*

If A is a **fractional ideal** *of k, then A^* denotes the* **complementary ideal** *to A with respect to the trace tr_k, defined as*

$$A^* \ := \ \{x \in k : tr_k(xA) \subset R_k\}, \tag{2.49}$$

(cf. [L, II.1]). If $\{\alpha_1, ..., \alpha_n\}$ is a **basis** *of A over $\mathbb{Z}$, then $\{\alpha_1^*, ..., \alpha_n^*\}$, where $\{\alpha_i^*\}$ is the dual basis relative to the trace tr_k, is a basis of A^*.*

One of the main notions of algebraic number theory is the **different** *$D_{k/\mathbb{Q}}$. The different $D_{k/\mathbb{Q}}$ "differs" A^{-1} from A^*, i.e. cf.e.g. [K], [L] and [N]*

$$A^* \ = \ (D_{k/\mathbb{Q}}A)^{-1}. \tag{2.50}$$

One can show that

$$D_{k/\mathbb{Q}} \ := \ R_k^*. \tag{2.51}$$

The second main important notion is the **discriminant** *$d(k)$ of an algebraic number field.*

If $\{C_j\}$ are embeddings as considered above and $\{\alpha_j\}$ forms a base of a fractional ideal A, then we can define the **discriminant** *$d_k(\alpha_1, ..., \alpha_n)$ by*

$$d_k(\alpha_1, ..., \alpha_n) \ := \ (det[C_j(\alpha_i)]_{i,j})^2 = det[tr_k(\alpha_i\alpha_j)]. \tag{2.52}$$

It is well-known that the discriminant of a basis of A does not depend on the choice of this basis. In particular, if $A = R_k$, then this discriminant is called the **discriminant of the field** *k and denoted by $d(k)$. The discriminant $d(k)$ has many nice and important properties :*

(1) according to the **Stickelberger theorem***, $d(k)$ is either congruent to unity (mod 4) or is divisible by 4,*

(2) is strictly connected with the signature $r = [r_1, r_2]$: $sign d(k) = (-1)^{r_2}$, and according to the **Minkowski theorem**

$$| d(k) | > (\frac{\pi}{4})^{2r_2}(\frac{n^n}{n!})^2,$$

which also ilustates the strict relation with the degree $n = n(k)$.

(3) The connection with the different :

$$N(D_{k/\mathbb{Q}}) = | d(k) | .$$

However the value of $d(k)$ is mainly underlined by the deep **Hermite theorem***, which asserts that only a finite number of algebraic fields can have the same discriminant.*

Besides the importance of $d(k)$, its arithmetic invariance does not appear in our "topological" generalization of HRace.

We saw that one of the main roles in the proof of (Face) was played by the function $\omega_{C(A)}$. In Lang's proof of HRace [L,XII.3], this corresponds to the consideration of the gaussian fixed point $\omega := \otimes_{j=1}^{r_1} G_1 \otimes_{j=1}^{r_2} G_{\mathbb{C}}$, where

$$G_{\mathbb{C}}(z) := e^{-\pi|z|^2} ; \; z \in \mathbb{C},$$

is the **complex Gaussian fixed point** *of $\mathcal{F}_2$ on $\mathbb{C}(= \mathbb{R}^2)$.*

Then

$$\omega(C(A)x^t) = exp(-\pi(N(A)^2 d(k))^{-1/n} \sum_{j=1}^{n} | \sum_{v=1}^{n} C_v(\alpha_j)x_j |^2) =: exp(-\pi(N(A)d(k))^{-1/n} < A_\alpha x, x >),$$

where the $\nu\mu$-component of the matrix $A_\alpha = [a_{\nu\mu}]$ is given by

$$a_{\nu\mu} := \sum_{j=1}^{n} C_j(\alpha_\nu \alpha_\mu),$$

and $< .,. >$ is the standard scalar product.

The matrix A_α is a **symmetric positive definite** *matrix. We can thus write*

$$A_\alpha = B_\alpha^2,$$

for some **symmetric matrix** *B_α. Therefore, ($B_\alpha^* = B_\alpha$)*

$$< A_\alpha x, x >=< B_\alpha^2 x, x >=< B_\alpha x, B_\alpha^* x >=|| B_\alpha x ||^{2_n}$$

and

$$exp(-\pi(N(A)^2 d(k))^{-1/n} < A_\alpha x, x >) = exp(-\pi(N(A)^2 d(k))^{-1/n} \mid\mid B_\alpha x \mid\mid_n^2) = \omega_{B_\alpha}(x)\ ; x \in \mathbb{R}^n.$$

Thus, $C(A)$ **corresponds** *to* B_α *in Lang's considerations of this gaussian fixed point. From [L, III.1] it is immediately follows that the inverse matrix of* A_α *is given by*

$$< A_\alpha^{-1} x, x > = \sum_{j=1}^{n} \mid \sum_{v=1}^{n} C_j(\alpha_v^*) x_v \mid^2 . \tag{2.53}$$

Furthermore, the absolute value of the discriminant is

$$\mid D_k(\alpha_1, ..., \alpha_n) \mid = \ det(A_\alpha).$$

One can establish the value of $H_r^0(E(k))$ *exactly in the same way as in [L, XIII.3]. More exactly, it is not difficult to calculate that*

$$H_r^0(E(k)) \ = \ 2^{r_1 - 1} R(k), \tag{2.54}$$

where $R(k)$ *is the* **regulator** *of* k *defined as follows : let* $u_1, ..., u_{r_1+r_2}$ *be* **independent generators** *for the unit group* $U(k)$ *(modulo roots of unity) (the* **Dirichlet's theorem***). The absolute value of the determinant*

$$det[N_v log \mid C_j(u_v) \mid] \tag{2.55}$$

(here N_v *- as usual - denotes a local degree) is independent of the choice of our generators* $\{u_j\}$ *and is called the* **regulator** $R(k)$ *of the field* k*. We note that this regulator, like all determinants, can be interpreted as a volume of a parallelotope in* $(r_1 + r_2)$*-space.*

Finally, the zeta function $\zeta_k(s)$ *has a simple pole at* $s = 1$ *with a* **residue** *equal to*

$$\frac{2^{r_1} (2\pi)^{r_2} h(k) R(k)}{w(k) \mid d(k) \mid}$$

and the non-zero constant λ_k *in the* **zero-polar factor** *(trivial zeta) from the Face theorem*

$$\lambda_k \ = \ \frac{2^{r_1} h(k) R(k)}{w(k)}. \tag{2.56}$$

3 RH-fixed points of $\mathcal{F}_n$

In this section we present constructions which lead to the derivation of the main technical tool of this paper - the **harmonic notion** of an RH-fixed point of the n-dimensional real Fourier transform. We present here a more abstract and brief version of the technique which was originally developed in $[M_A]$ for the proof of the Riemann hypothesis.

Let V be a **real vector space** endowed with an **idempotent endomorphism** $F : V \longrightarrow V$, i.e. $F^2 = I_V$, where I_V denotes the identity endomorphism of V.

Let us consider the **purely algebraic** notion of the **quasi-fixed point of** F associated with a parameter $l \in \mathbb{C}$ and an element $v \in V$:

$$Q_l(F)(v) \;=\; Q_l(v) \;:=\; v \;+\; lF(v). \tag{3.57}$$

Let us observe that if $l = 1$ then $Q_1(v)$ is a **fixed point** of F, i.e.

$$F(Q_1(v)) = F(v) + F^2(v) = F(v) + v = Q_1(v), \tag{3.58}$$

and if $l = -1$ then $Q_{-1}(v)$ is a (-)fixed point of F, i.e.

$$F(Q_{-1}(v)) = F(v) \;-\; F^2(v) \;=\; -(v \;-\; F(v)) = -Q_{-1}(v).$$

We obtain the following result on the existence of quasi-fixed points

Lemma 2 *(***Existence of quasi-fixed points***).*
For each $v_0 \in V$ and $l \neq \pm 1$ the **formula**

$$v_l \;:=\; \frac{v_0}{1 - l^2} \;-\; \frac{lF(v_0)}{1 - l^2} \tag{3.59}$$

gives the solution of the following **Abstract Fox Equation***(AFE in short , cf. also $[M_A]$)*

$$(AFE_V^l) \;\; v_l \;+\; lF(v_l) \;=\; v_0. \tag{3.60}$$

Lemma 2 shows that making a simple algebraic calculus, we cannot obtain a **singular** solutions of AFE_V, since the formula (3.59) **has no sense** for $l = \pm 1$.

Moreover, we see that on the ground of **classical logic** the $\pm$ fixed point $Q_{\pm 1}(v_0)$ cannot be the solution of $(AFE_V^{\pm})$, $Q_{\pm}(v_{\pm}) = v_0$ if v_0 is not a $\pm$fixed point of F.

Let us denote the real subspace of V of all $\pm$fixed points v_0 of F in V by $Fix_{\pm}(F)$, i.e. $F(v_0) = \pm v_0$. We thus see that the condition

$$v_0 \;\in\; Fix_{\pm}(F) \tag{3.61}$$

is a **necessary condition** for the **existence** of solutions $Q_{\pm 1}(v_0)$ of (AFE_V).

We construct $Q_{\pm 1}(v_0)$ using the **averaging procedure** for the family $\{Q_l(v_0) : l \neq \pm 1\}$, originally constructed in $[M_A]$.

As in $[M_A]$, it will be very convenient to use the unique non-commutative field of **Hamilton quaternions** $\mathbb{H}$ (the **Einstein space-time space**).

We denote a **Haar measure** of the additive group $(\mathbb{H}, +)$, by $H_{\mathbb{H}}$, i.e. the **standard Lebesgue measure** d^4h of the vector space $\mathbb{R}^4$ (the Einstein space-time).

For each $M, N > 0$ we consider the **hamiltonian segments** (rings)

$$S(M, N) \;:=\; \{h \in \mathbb{H} : M \leq \mid h \mid_{\mathbb{H}} \leq N\}, \tag{3.62}$$

where in all the sequel $|\cdot|_{\mathbb{H}} := ||\cdot||_4$ is the standard Euclidean norm on $\mathbb{R}^4$.

Finally, we consider the **invertion** $I_{\mathbb{H}}$ of $\mathbb{H}$

$$I_{\mathbb{H}}(l) \ := \ l^{-1} \ , \ l \in \mathbb{H}^* := \mathbb{H} - \{0\}. \tag{3.63}$$

Let us observe that $I_{\mathbb{H}}$ is only a set-automorphism (and not a group automorphism of the multiplicative group $\mathbb{H}^*$, since it is not commutative).

Each **automorphism** λ of $(\mathbb{H}, +)$ changes the Haar measure $H_{\mathbb{H}}$ into $cH_{\mathbb{H}}$ with $c \in \mathbb{R}^*_+$ (the **von Neumann-Weil theorem**). The number c does not depend on the choice of Haar measure. It is denoted by $\Delta_{\mathbb{H}}(\lambda)$ and is called the **Haar module** of λ. It is defined by any of the equivalent formulas given below (cf.[W,I])

$$(W_m) \ \ H_{\mathbb{H}}(\lambda(B)) \ = \ \Delta_{\mathbb{H}}(\lambda)H_{\mathbb{H}}(B)$$

or

$$(W_i) \ \ \int f(\lambda^{-1}(x))dH_{\mathbb{H}}(x) = \Delta_{\mathbb{H}}(\lambda)\int f(x)dH_{\mathbb{H}},$$

where B is any Borel set and f is any integrable function with $\int f dH_{\mathbb{H}} \neq 0$.

The second formula can be symbolically written in the form:

$$dH_{\mathbb{H}}(\lambda(x)) \ = \ \Delta_{\mathbb{H}}(\lambda)dH_{\mathbb{H}}(x).$$

If $h \in \mathbb{H}^*$ is arbitrary, then the formula : $M_h(x) := h \cdot x, x \in \mathbb{H}$ defines a linear multiplication automorphism of $(\mathbb{H}, +)$. We set

$$\Delta_{\mathbb{H}}(h) \ := \ \Delta_{\mathbb{H}}(M_h) \ , \ h \in \mathbb{H}^*,$$

and moreover, we define $\Delta_{\mathbb{H}}(0) := 0$. It is well-known that (cf.e.g. [W, I.2 and Corrolary 2])

$$\Delta_{\mathbb{H}}(h) \ = | \ h \ |^4_{\mathbb{H}} = || \ h \ ||^4_4 \ . \tag{3.64}$$

We denote the **invertion** of $\mathbb{H}^*$ by $I_{\mathbb{H}}(h) := h^{-1}, h \in \mathbb{H}^*$. Unfortunately, $I_{\mathbb{H}} = I$ is not a group automorphism of $\mathbb{H}^*$, since $\mathbb{H}^*$ is not commutative! However, it is still a very **crucial topologically-algebraic** map of $\mathbb{H}^*$ of order 2 : $I^2_{\mathbb{H}} = id_{\mathbb{H}}$.

Thus, beside such an important invariant of $\mathbb{H}$ like the Galois group $Gal(\mathbb{H}/\mathbb{R})$, we have an additional important **invariant** of $\mathbb{H}$ - the **invertion group** $Inv(\mathbb{H}^*) := \{id_{\mathbb{H}^*}, I_{\mathbb{H}^*}\}$ of $\mathbb{H}^*$ (cf. $[M_A]$).

It is well-known (cf.e.g.$[M_A$, Lem.4]) that

$$dH_{\mathbb{H}^*}(h) \ := \ \frac{dH_{\mathbb{H}}(h)}{| \ h \ |^4_{\mathbb{H}}} \ , \ h \in \mathbb{H}^*. \tag{3.65}$$

is a (left) **Haar measure** of the multiplicative group $\mathbb{H}^*$. Moreover, it would be convenient to recall the **simple algebraic -measure formulas** for $H_{\mathbb{H}}$ and $H_{\mathbb{H}^*}$ given below (cf. M_A, Prop.3) : for each **integrable** function f on $\mathbb{H}^*$ we have :

$$\int_{\mathbb{H}^*} f(h^{-1})dH_{\mathbb{H}^*}(h) \ = \ \int_{\mathbb{H}^*} f(h)dH_{\mathbb{H}^*}(h). \tag{3.66}$$

i.e. $H_{\mathbb{H}^*}$ is the **invariant measure** (or the **Bogoluboff-Kriloff measure**) of the **dynamical system** $(\mathbb{H}^*, I_{\mathbb{H}})$. Moreover,

$$\int_{\mathbb{H}^*} f(h) dH_{\mathbb{H}} \;=\; \int_{\mathbb{H}^*} \frac{f(h^{-1}) dH_{\mathbb{H}}(h)}{\mid h \mid_{\mathbb{H}}^{8}}. \tag{3.67}$$

For each $N > M > 0$ we consider the **compact** $\mathbb{H}$**-rings**

$$R_{\mathbb{H}}(M, N) \;:=\; \{h \in \mathbb{H} : M \leq \mid h \mid_{\mathbb{H}} \leq N\}, \tag{3.68}$$

and the corresponding **dynamical sub-system** of $(\mathbb{H}^*, I_{\mathbb{H}})$

$$D_{\mathbb{H}}(M, N) \;:=\; (R_{\mathbb{H}}(M, N) \cup R_{\mathbb{H}}(N^{-1}, M^{-1}), I_{\mathbb{H}}), \tag{3.69}$$

with $M, N > 1$.

From (3.66) we immediately obtain that the formula

$$\beta_{\mathbb{H}}(A) \;:=\; \int_A \frac{d^4 h}{\mid 1 - h^2 \mid^4} \;;\; h \in D_{\mathbb{H}}(M, N), \tag{3.70}$$

gives an **invariant measure** of $D_{\mathbb{H}}(M, N)$ - the **Herbrand distribution** of $\mathbb{H}^*$ (cf.[M_A]). In particular, the measure $\beta_{\mathbb{H}}$ satisfies the condition

$$\beta_{\mathbb{H}}(I_{\mathbb{H}}^{-1}(A)) \;=\; \beta_{\mathbb{H}}(A). \tag{3.71}$$

We use below the theory of the **sextet** $(\mathbb{H}, \Delta_{\mathbb{H}}, H_{\mathbb{H}}, R_{\mathbb{H}}(M, N), \beta_{\mathbb{H}}, R_{\mathbb{H}})$.

For the sake of completness, we also briefly recall here two deep and difficult results from **analytic potential theory** of $\mathbb{H}$ explored in $[M_A]$:

(1) **The Riesz theorem** (cf.[HK, Sect.3.5, Th.3.9]).

Let $s = s(x)$ be a **subharmonic** function in a domain of $\mathbb{R}^6$. Then there exists a **hamiltonian Riesz measure** $R_{\mathbb{H}}$ and a **harmonic function** $h(x)$ outside a compact set E, such that

$$(RT)\; s(x) \;=\; \int_E \frac{dR_{\mathbb{H}}(y)}{\mid\mid x - y \mid\mid_6^4} \;+\; h(x) \;;\; x \in \mathbb{R}^6.$$

(2) **Brelot's theorem** (cf.[HK, Sec.36, Th3.10] - on the existence of **harmonic measures**).

Let D be a **regular** and **bounded domain** of $\mathbb{R}^n$ with border ∂D. Then, for each $x \in D$ and arbitrary Borel set B of ∂D , there exists a **unique number** $\omega(x, B : D)$, which is a **harmonic function in** x and **probabilty measure in** B and moreover, for each **semicontinuous function** $f(\xi)$ on ∂D the formula

$$(DP)\; \tilde{f}(x) \;=\; \int_{\partial D} f(\xi) d\omega(x, \xi; D) \;;\; x \in D - \partial D, \tag{3.72}$$

gives the **harmonic extension** of f from ∂D to D.

The family of harmonic measures $\omega(D) := \{\omega(x, \cdot; D) : x \in D\}$ solves the **Dirichlet problem**(DP) for a pair $(D, \partial D)$ and if a solution exists it is **unique**.

In $[M_A]$ we introduced the following formal definition of the **Abstract Hodge Decomposition**: let $f : X \longrightarrow \mathbb{C}$ be a function and $K : X \times I \longrightarrow \mathbb{C}$ another "kernel" function. A measure H_f on a σ-field of subsets of I gives the Abstract Hodge Decomposition of f, if the following integral representation is satisfied

$$(AHD_f)\ f(x) \ = \ \int_I K(x,i) dH_f(i) \ ; \ x \in X.$$

We call the measure H_f, which appeares in (AHD_f) the **Hodge measure** of f.

Proposition 1 *(***Existence of** *$AHD_{\mathbb{H}}$).*

There exists such a Borel probability measure $R_{\mathbb{H}}$ (a **hamiltonian Riesz measure***) on the 3-dimensional sphere $S^3 := \{h \in \mathbb{H} :\mid h \mid_{\mathbb{H}} = 1\}$, such that for each*

$$r \in X_{\mathbb{H}}(M,N) \ := \ R_{\mathbb{H}}(M,N) \cup R_{\mathbb{H}}(N^{-1}, M^{-1})$$

with $N > M > 1$, the following abstract Hodge decomposition ($AHD_{\mathbb{H}}$ in short) holds :

$$(AHD_{\mathbb{H}})\ \Delta_{\mathbb{H}}^{-1}(r^2) \ = \ \int_{S^3} \frac{dR_{\mathbb{H}}(h)}{\Delta_{\mathbb{H}}(r^2 - h^2)} \ = \ \mathcal{H}_{\mathbb{H}}(R_{\mathbb{H}})(r).$$

Proof. Let $\epsilon_n > 0$ be an arbitrary sequence, which converges to zero. Then the functions $|| \cdot ||_6^{-(4+\epsilon_n)}$ are **subharmonic** (as suitable powers of a **harmonic function**) and obviously they are **not harmonic**! Therefore, according the **Riesz theorem**, there exists a sequence of **Riesz measures** $\{R_n\}$ and a sequence $\{h_n\}$ of harmonic functions **inside** of S^3 with the property

$$|| \ r \ ||_6^{-(4+\epsilon_n)} = \ \int_{S^3} \frac{dR_n(h)}{|| \ r - h \ ||_6^4} \ + \ h_n(r). \tag{3.73}$$

Since $dR_n(x) \ = \ \nabla(|| \ x \ ||_6^{(4+\epsilon_n)})dx$ (cf. [HK, Section 3.5]) , the sequence $\{R_n(S^3)\}$ is **bounded**, i.e. $R_n(S^3) \leq A$, for some $A > 0$ and all $n \in \mathbb{N}$.

According to **Frostman's theorem** (cf. [HK, Theorem 5.3]), we can choose a subsequence $\{R_{n_p}\}$, which is **weakly convergent** to a limit measure R_∞ on S^3, i.e. $R_\infty := (w) \lim_{p \longrightarrow \infty} R_{n_p}$ and

$$|| \ r \ ||_6^{-4} = \ \int_{S^3} \frac{dR_\infty(x)}{|| \ r - x \ ||_6^4} \ + \ h(r). \tag{3.74}$$

On the other hand, according to **Brelot's theorem** applied to the triplet $(B^6(1), 0, S^5)$ - there exists a **harmonic measure** $\omega(\cdot) := \omega(\cdot, 0, B_6)$ on S^5 with the property

$$|| \ r \ ||_6^{-4} = \ \int_{S^5} \frac{d\omega(y)}{|| \ r - y \ ||_6^4} \ , \ r \in (B^6)^c. \tag{3.75}$$

Let us denote the natural inclusion by $j_{35}: S^3 \longrightarrow S^5$; $j_{35}(h) = (h,0,0)$. Then (3.74) can be written of the form

$$|| r ||_6^{-4} = \int_{S^5} \frac{d(j_{35}^* R_\infty)(y)}{|| y - r ||_6^4} + h(r). \tag{3.76}$$

Let us consider the **continuous function** $f(\xi) := || \xi ||_6^{-4}$ on S^5 and (for a while) take μ to be one of the two measures : $j_{35}^*(R_\infty)$ or ω. Finally, let us consider the potential $\int_{S^5} \frac{d\mu(y)}{||r-y||_6^4}$. The vectors $r, y \in \mathbb{R}^6$ can be considered as **Cayley numbers** from $\mathbb{R}^8 = \mathbb{H} \times \mathbb{H}$ and $|| \cdot ||_6$ as the restriction of the **Cayley norm**. Since Cayley numbers form a non-commutative and non-associative algebra with **division**, then we can write

$$\int_{S^5} \frac{d\mu(y)}{|| r - y ||_6^4} = \int_{S^5} \frac{d\mu(y)}{|| y(1 - r/y) ||_6^4} =: \int_{S^5} \frac{d\nu(r,y)}{|| y ||_6^4} = \int_{S^5} f(\xi) d\nu(r,\xi).$$

Thus, both the formulas (3.75) and (3.76) give the solution of the **Dirichlet problem** for $((B^6)^c, S^6, f)$. From the **uniqueness** of the solution of the Dirichlet problem (cf. [HK, Th.1.13]), we obtain:

$$j_{35}^*(R_\infty) = \omega \;\; and \;\; h \equiv 0 \; on \; (B^6)^c,$$

since h, as a difference between a harmonic function and a potential, is also harmonic on $\mathbb{R}^6 - S^5$. Hence, restricting ourselves in (3.76) for $r = h \in \mathbb{H}$, we finally obtain

$$\Delta_{\mathbb{H}}(h^{-1}) = \int_{S^3} \frac{dR_\infty(x)}{\Delta_{\mathbb{H}}(h - x)}, \; h \in S_{\mathbb{H}}(M,N). \tag{3.77}$$

Let us consider a **branch of the hamiltonian square root** $\sqrt{\cdot}$ and the induced map of measure spaces : $\sqrt{\cdot} : (S^3, R_\infty) \longrightarrow (S^3, \sqrt{\cdot}^* R_\infty)$, substituting h^2 for h and $R_{\mathbb{H}}$ for $\sqrt{\cdot}^* R_\infty$ we obtain the above proposition.

We will use the **hamiltonian sextet** (from analytic potential theory)

$$(\mathbb{H}, |\cdot|_{\mathbb{H}}, \Delta_{\mathbb{H}}, H_{\mathbb{H}}, H_{\mathbb{H}^*}, R_{\mathbb{H}}, \beta_{\mathbb{H}})$$

in the averaging procedure given below to obtain, singular solutions of (AFE_V).

A similar result is much easier to obtain using the completely different nature of locally compact rings - the small adeles , i.e. working with the **adic potential theory**.

As we will show below, in the p-adic case, the required **algebraic** potential theory is simpler, in opposite to the strongly analytic potential theory of $\mathbb{R}^m$. Therefore, the p-adic fields (and generally local non-archimedean fields are - in such a way we see them today - are missing links - to the needed maths constructions).

Let H_p denotes the **Haar measure** of the additive group of the p-adic number field $\mathbb{Q}_p$. The main reason that the algebraic potential theory over $\mathbb{Q}_p$ is simpler that the analytic one over $\mathbb{R}^m$ is the quite different behaviour of Haar measures on totally - disconnected fields with compare to Haar measures on the connected fields. In particular, $\mathbb{Z}_p^*$ is **open**, and therefore $H_p(\mathbb{Z}_p^*) \neq 0$, whereas in the case of **connected** local fields K we have

$$H_K(S_K) = 0,$$

where S_K is the unit sphere in K.

Thus, it is convenient to normalizeed H_p in such a manner that

$$H_p(\mathbb{Z}_p^* = S_p) \ = \ (1 \ - \ p^{-1}),$$

(the Euler component in $\zeta_{\mathbb{Q}}^{-1}(1)$)

Let p and q be two different **prime numbers** and $\mathbb{Q}_p$ and $\mathbb{Q}_q$ be the fields of p-adic and q-adic numbers. For the convenience we take the non-canonical choice of $| \ . \ |_q$ by putting $| \ q \ |_q = 1/p$!

We denote by $\mathbb{Q}_{pq}$ the product $\mathbb{Q}_p \times \mathbb{Q}_q$, being the locally compact abelian ring with a large set of invertible elements (its completion has Haar measure zero). We denote its **Haar module** by Δ_{pq}, and its **Haar measure** by H_{pq}. R_{pq} is the **adic Riesz measure**, I_{pq} is the **invertion automorphism** of $\mathbb{Q}_{pq}^*$ and finally, we denote the **adic Herbrandt distribution** (the Bogoluboff-Krilov measure) of $\mathbb{Q}_{pq}$ by β_{pq}. Then the sextet

$$(\mathbb{Q}_{pq}, \mathbb{Q}_{pq}^*, \Delta_{pq}, H_{pq}, I_{pq}, \beta_{pq})$$

enables us to show, in a relatively simple, algebraic way, the existence of the below pq**-adic Abstract Hodge Decomposition**((AHD_{pq}) in short).

According to the Weil's Lemma 2 (see [W, I.2., Lemma 2]) we have

$$\Delta_{pq}(x) \ = \ mod_{\mathbb{Q}_{pq}}(x) \ = | \ x_p \ |_p | \ x_q \ |_q \ , \ x = (x_p, x_q) \in \mathbb{Q}_{pq}^*.$$

We define a **sub-dynamical system** $D_{pq}(M,N) = (X_{pq}(M,N), I_{pq})$ of the dynamical system of the small adeles $(\mathbb{Q}_{pq}, I_{pq})$. Moreover, in the sequel we simply write D_{pq} and X_{pq} instead of $D_{pq}(M,N)$ and $X_{pq}(M,N)$, respectively. The compact topological space X_{pq} is defined as follows: let $M, N \in \mathbb{N}^*$ be such that $1 \leq M < N$. Then

$$X_{pq}(M,N) \ := \ \{x \in \mathbb{Q}_p \times \mathbb{Q}_q : x = (x_p, x_q), | \ x_q \ |_q = 1, | \ x_p \ |_p \in [p^{-N+1}, p^{-M}] \cap [p^M, p^{N-1}] == \ I_{\mathbb{R}}([p^M, p^{N-1}]) \cap [p$$

Finally, let us consider the the p-dic projection $P_p : \mathbb{Q}_{pq} \longrightarrow \mathbb{Q}_p, P_p(x_p, x_q) = x_p$ and I_p**-invariant** function

$$\mathcal{I}_p(\lambda) \ := \ \frac{| \ \lambda \ |_p}{| \ 1 \ - \ \lambda^2 \ |_p} \ , \ \lambda \in \mathbb{Q}_p^* - \{1\}.$$

Under the above notations we have

Lemma 3 *(***On the pq-adic Herbrandt measure** *$d\beta_{pq}$).*

The formula

$$d\beta_{pq}(x_p, x_q) := \frac{d(H_p \times H_q)(x_p, x_q)}{| \ 1 \ - \ x_p^2 \ |_p} = \frac{\mathcal{I}_p(P_p(x_p, x_q)) d(H_p \times H_q)(x_p, x_q)}{\Delta_{pq}((x_p, x_q))}$$

gives a **Bogoluboff-Kriloff measure***(* **Herbrandt distribution***) of D_{pq}.*

Proof. Since Δ_{pq} is the Haar module of $\mathbb{Q}_p$ (like $\Delta_{k_A^*}(z) = \prod_{v \in P} \mid z_v \mid_v$) in the case of **ideles** (see e.g. [W] and [Ko]), then the equality

$$\Delta_{pq}(x_p, x_q) \ = \mid x_p \mid_p,$$

holds on $X_{pq}(M, N)$. Hence we get

$$\frac{d(H_p \times H_q)(x_p, x_q)}{\mid 1 - x_p^2 \mid_p} = \frac{\mid x_p \mid_p}{\mid 1 - x_p^2 \mid_p} \cdot \frac{d(H_p \times H_q)(x_p, x_q)}{\mid x_p \mid_p} =$$

$$\mathcal{I}_p(P_p(x_p, x_q)) \cdot \frac{d(H_p \times H_q)(x_p, x_q)}{\Delta_{pq}(x_p, x_q)}.$$

Since $\frac{d(H_p \times H_q)}{\Delta_{pq}}$ is a Bogoluboff-Kriloff measure of $(\mathbb{Q}_{pq}^*, I_{pq}), P_p \circ I_{pq} = I_p \circ P_p$ and $\mathcal{I}_p$ is I_p-invariant, that we really see that the above formula gives a pq-adic Bogoluboff-Kriloff measure of D_{pq}. (Let us remark the importance of the fact that $H_q(\mathbb{Z}_q^*) \neq 0$).

Proposition 2 *(***The existence of** *AHD_{pq}).*

$$(AHD_{pq}) \ \ \Delta_{pq}(x^{-2}) \ = \ \int_{S_{pq}} \frac{dR_{pq}(y)}{\Delta_{pq}(x^2 - y^2)} \ \ if \ x \in X_{pq}(M, N),$$

where $S_{pq} := \{x \in \mathbb{Q}_{pq} : \Delta_{pq} = 1\}$ is the unit adic sphere.

Proof. Let $x \in X_{pq}(M, N)$ be arbitrary. Then $x = (x_p, x_q)$ with $\mid x_q \mid_q = 1$ and therefore $\Delta_{pq}(x) = \mid x_p \mid_p$ (we can identity the p-adic field $\mathbb{Q}_p$ with the subset $\mathbb{Q}_p \times \{1\}$ of $\mathbb{Q}_{pq}$). But $\mid x_p \mid_p \geq p^{-N+1}$, and therefore according to the **ultrametricity** of $\mid \cdot \mid_p$ (see e.g. [W, I.2., Corrolary 4]) for all $y \in \mathbb{Q}_p$ with $\mid y \mid_p = p^{-N}$ we have

$$\mid x_p^2 \mid_p = \mid x_p^2 \ - \ y^2 \mid_p .$$

Integrating the both sides of the inverse of the above equality with respect to the Haar measure H_p on $p^N \mathbb{Z}_p^*$, for each $\eta \in \mathbb{Q}$ with $\mid \eta \mid_p = 1$, we obtain :

$$\frac{1}{\mid x_p^2 \mid_p} \ = \ \frac{1}{H_p(p^N \mathbb{Z}_p^*)} \int_{p^N \mathbb{Z}_p^*} \frac{dH_p(\xi)}{\mid x_p^2 \ - \ (\eta \xi)^2 \mid_p}.$$

Let us denote : $d\nu_p(\xi) := \frac{dH_p(\xi)}{H_p(p^N \mathbb{Z}_p^*)}$. Then, for all $\eta \in \mathbb{Q}$ with $\mid \eta \mid_p = 1$, the above equality can be written as

$$\frac{1}{\mid x_p^2 \mid_p} \ = \ \int_{p^N \mathbb{Z}_p^*} \frac{d\nu_p(\xi)}{\mid x_p^2 \ - \ (\eta \xi)^2 \mid_p}.$$

(we non-standartly assumed that $\mid q \mid_q = p^{-1}$).

Let F be any **finite** subset of $\{\eta \in \mathbb{Q} :\mid \eta \mid_p = 1, \mid \eta \mid_q = p^{-N}\} \subset p^{-N}\mathbb{Z}_q^*$. For an arbitrary subset A of $\mathbb{Q}$ we define the measure μ_q^F by

$$\mu_q^F(A) \ := \ \sum_{f \in F} \frac{\delta_f(A)}{\# F},$$

where δ_f is the Dirac measure at f.

Let us consider the measure $(\nu_p \times \mu_q^F)$. Summing the both sides of the previous measure representation on F we obtain

$$\frac{1}{\mid x_p^2 \mid_p} \ = \ \int\int_{p^N \mathbb{Z}_p^* \times p^{-N}\mathbb{Z}_q^* \subset S_{pq}} \frac{d(\nu_p \times \mu_q^F)(\xi, \eta)}{\mid x_p^2 \ - \ (\eta\xi)^2 \mid_p}.$$

Let us look at the natural inclussion $j_{pq} \ : \ p^N\mathbb{Z}_p^* \times p^{-N}\mathbb{Z}_q^*$ as on a **random variable** $j_{pq} : (p^N\mathbb{Z}_p^* \times p^{-N}\mathbb{Z}_q^*, \nu_p \times \mu_q^F) \longrightarrow S_{pq}$. Then the **distribution** R_{pq} of j_{pq} we will be called the **(p,q)-adic Riesz measure** and the right-hand side of the above formula we can finally write in the form :

$$\frac{1}{\mid x_p^2 \mid_p} \ = \ \int_{S_{pq}} \frac{R_{pq}(dy)}{\mid x_p^2 \ - \ P_p(y)^2 \mid_p}.$$

Combining the above formulas, we obtain the proof of the existence of the (AHD_{pq}). It also shows that the proof of (AHD_{pq}) is possible in a completely **algebraic way**.

Remark 2 *Probably the first mathematician, who considered and applied the p-adic potential theory was* **Kochubei**. *In the case of p-adic fields $\mathbb{Q}_p$, the $\mathbb{Q}_p$-Hilbert transforms probably first were considered in the* **Vladimirov** *et al.'s paper [VWZ] as the γ-order* **derivative** *$D^\gamma f$ of a locally constant function f. It is describable by pseudo-differential operator and explicitly written as*

$$D^\gamma f(x) = \int_{\mathbb{Q}_p} \mid \xi \mid_p^\gamma \hat{f}(\xi)\chi_p(-\xi x)H_p(d\xi) = \frac{p^{\gamma-1}}{1-p^{-\gamma-1}} \int_{\mathbb{Q}_p} \frac{f(x)-f(y)}{\mid x-y \mid_p^{\gamma+1}} H_p(dy),$$

where χ_p is the additive character of $\mathbb{Q}_p$ and $\hat{f}(\xi)$ stand for the Fourier transformation $\int_{\mathbb{Q}_p} \chi_p(\xi x) f(x) H_p(dx)$ of a function f. A deeper analysis of p-adic fractional differentiation D^γ is given in the Kochubei's book [Ka], where using **Minlos-Mądrecki's theorem**, *he established the existence of a* **Kochubei-Gauss measure** *μ over infinite-dimensional field extensions Ω_p of $\mathbb{Q}_p$, which is a* **harmonic measure** *for D^γ and solves* **p-adic integral equations of a profile of wing of a plane** *in the case of Ω_p (see [Ka, Prop.6]).*

The importance of R_{pq} is also underline by the fact that unfortunately, firstly we have the following negative result concerning (AHD_{pq}).

Non-existence of solutions of p-adic profile of a wing in functions

Let p be an arbitrary prime number. There is not exist an **absolutely continuous measure** h_p *w.r.t. the Haar measure* H_p *(the p-adic harmonic measure), which gives the following p-adic Abstract Hodge Decomposition (cf. [M_A]) with the property*

$$(AHD_p)\ \frac{1}{\mid x\mid_p^2}\ =\ \int_{\mathbb{Z}_p^*}\frac{dh_p(y)}{\mid x^2-y^2\mid_p}\ ,\ x\in S_p(M,N).$$

Proof. *The proof is based on the remarkable property of the Haar measure* $H_p : H_p(\mathbb{Z}_p^*) = 1-p^{-1}$ *(the* **Euler factor in the Riemann zeta***). Assume (a contrary), that there exists a measure* h_p*, which is absolutely continuous w.r.t.* $H_p : h_p \ll H_p$*. Let us denote its density by* ω_p*, i.e.*

$$\omega_p\ =\ \frac{dh_p}{dH_p}.$$

This conjecture permits us to apply the big and well-known machinery of p-adic Fourier analysis to the problem of the existence of (AHD_p)*. Reely, the* (AHD_p) *is obviously equivalent to the formula*

$$\chi_{S_p(M,N)}(x)\mid x\mid_p^{-1}=\int_{\mathbb{Q}_p}\mid x-y\mid_p^{-1}\chi_{\mathbb{Z}_p^*}(y)\omega_p(y)dH_p(y)\ =$$

$$=\int_{\mathbb{Q}_p}(\chi_{\mathbb{Z}_p^*}\cdot\omega_p)(x-z)\mid z\mid_p^{-1}dH_p(z):=[(\chi_{\mathbb{Z}_p^*}*\mid\cdot\mid_p^{-1})](x)\ ,\ x\in\mathbb{Q}_p,$$

where χ_A *denotes the characteristic function of a set A and* $*$ *means the p-adic* **convolution**.

If we apply the p-adic Fourier transform $\mathcal{F}_p$ *to the both sides of the above equalities, then we obtain*

$$(\chi_{S_p(M,N)}\hat{\cdot}\mid\cdot\mid_p^{-1})(\xi)\ =\ \mid\cdot\hat{\mid}_p^{-1}(\xi)\cdot\chi_{\mathbb{Z}_p^*}\hat{\cdot}\,\omega_p(\xi)\ ,\ \xi\in\mathbb{Q}_p.$$

Let us observe that

$$\mid\cdot\hat{\mid}_p^{-1}(\xi)\ =\ \frac{(1-p)log\mid\xi\mid_p}{plogp}\ ,\ \xi\in\mathbb{Q}_p,$$

see [Ka, Sect. 1.5, formula (1.29)]. Thus, the last equality is not possible.

In the light of the above presented negative result, the previous above result - on the existence of (AHD_{pq}) *- gathers a greater value.*

The small pq-adele ring $\mathbb{Q}_{pq}$ *is only one representant from a whole class of "models", of the very similar nature, which can be used in the same context.*

*(1)***The pq-adic vector space** $\mathbb{Q}_{[pq]}$.

It is well-known (see e.g. [La, Sect.1]) that the "world" of **valuations** *(or* **absolute values** *or* **points***) is very rich. In particular, we saw, how effective was the action of the defined below pre-valuations* v_{pq}*, which gives* (AHD_{pq}) *of the p-adic valuation* $\mid\cdot\mid_p$ *in the simple* **algebraic way***, if we compare it with a difficult* **analytic** *proof of* $(AHD_{\mathbb{H}})$ *of*

$\Delta_{\mathbb{H}}^{-2}$. *With a similar situation we have deal in the famous Faltings' proof of the Mordell-Shafarevich-Tate conjectures. He used so called* **heights of global fields**, *which are some functions defined by p-adic valuations (cf.e.g. [La, Fa]).*

According to the principal theorem of the arithmetics each non-zero rational number x we can uniquely write of the form:

$$x \;=\; \frac{a}{b}p^m q^n,$$

where $a, b, m, n \in \mathbb{Z}, (a,b) = 1$, and pq do not divide ab. For such a rational x we put

$$\alpha_p(x) \;:=\; m, \;\; \alpha_q(x) \;:=\; n,$$

and

$$v_{pq}(x) \;:=\; p^{-(m+n)} \;=\; p^{-(\alpha_p(x)+\alpha_q(x))}.$$

The functions $\alpha_p : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined below are called **exponents** *corresponding to p and satisfies few simple and nice elementary properties. A. Ostrowski showed their most surprising properties : they are* **unique** *arithmetical functions (up to a constant - like Haar measures), which satisfy the five mentioned above their elementary properties (see e.g. [Na1,Th.1.7(i)-(v)]). In particular, the functions v_{pq} satisfies the following condition :*

$$v_{pq}(x) \;= \mid x \mid_p \mid x \mid_q,$$

i.e. v_{pq} has only one good "residual" above **multiplicative** *property. Moreover*

$$v_{pq}(x+y) \leq max\{v_{pq}(x), v_{pq}(y), \mid x \mid_p \mid y \mid_p, \mid y \mid_p \mid x \mid_q\},$$

where $\mid \cdot \mid_p$ and $\mid \cdot \mid_q$ are p-adic and q-adic valuations, respectively, but with the additional assumption that

$$\mid q \mid_q = \; p^{-1}.$$

Thus, v_{pq} has a bad linear (ring) algebraic properties. In particular, $v_{pq}(\cdot)$ is not a **valuation** *but only - let us say - a* **pre-valuation**. *Therefore, the* **completion** *of $\mathbb{Q}$ w.r.t. the metric type function : $d_{pq}(x,y) \;=\; v_{pq}(x-y)$ is rather a pathological object and in particular, it is not a topological field.*

(2) By $\mathbb{Q}_{(pq)}$ we donote the set $\{0, 1, ..., pq-1\}(p,q)$ of all **double formal Laurent series** *with coefficients in the set $\{0, 1, ..., pq-1\}$. Thus, each element x of $\mathbb{Q}_{(pq)}$ has the form :*

$$x \;=\; \sum_{m=M}^{\infty}\sum_{n=N}^{\infty} a_{mn}p^m q^n \;;\; a_{mn} \; in\{0, 1, ..., pq-1\} \;,\; M, N \in \mathbb{Z}.$$

If we establish a (non-canonical) ordering $<_2$ on the lattice $\mathbb{Z}^2$, in such a way that $(\mathbb{Z}^2, \leq_2)$ and $(\mathbb{N}, \leq)$ are isomorphic in the category of ordered sets : $h : (\mathbb{Z}^2, \leq_2) \simeq (\mathbb{N}, \leq)$, and we establish the natural bijection

$$\mathbb{Q}_{(pq)} \supset x = \sum_{m=M}^{\infty}\sum_{n=N}^{\infty} a_{mn}p^m q^n \longrightarrow \sum_{n=h(M,N)} a_{h(m,n)}p^{h(m,n)} \in \mathbb{Q}_p \;=$$

$$= \sum_{n=h(M,N)} a_{\pi(h(m,n))} p^n,$$

(where here π denotes a respectible permutation of $\mathbb{N}$), then we can endow $\mathbb{Q}_{(pq)}$ with the natural local field structure (transformed from $\mathbb{Q}_p$). Thus $\mathbb{Q}_{(pq)}$ is a **local field** *isomorphic with $\mathbb{Q}_p$. Unfortunately, we cannot expect that $\mathbb{Z}_{(pq)}$ is isomorphic with $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}_q$, where $\mathbb{Z}_{(pq)} := \{x \in \mathbb{Q}_{(pq)} : v_{pq}(x) \leq 1\}$ (let us mention here that $\mathbb{F}_{pq} \neq \mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{F}_q = 0$).*

To see the compactness of $\mathbb{Z}_{(pq)}$ (and hence the local compactness of $\mathbb{Q}_{(pq)}$) it suffices to observe that the function f from the product D of a countable many copies of the pq-elements set $\{0, 1, ..., pq-1\}$ onto $\mathbb{Z}_{(pq)}$ given by

$$f(\{a_{mn}\}_{m,n=0}^{\infty}) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} p^m q^n$$

is **surjective** *and* **continuous** *in the Tichonov topology of D. Since D is compact then $\mathbb{Z}_{(pq)}$ is compact as a continuous image of the compact set.*

On the other hand, we have natural inclusions (in the category of sets): $i_p : \mathbb{Q}_p \longrightarrow \mathbb{Q}_{(pq)}$ and $i_q : \mathbb{Q}_q \longrightarrow \mathbb{Q}_{(pq)}$.

Observe however, that with the multiplications defined as :

$$\alpha \cdot_p x \ := \ i_p(\alpha) \cdot x \ \text{ and } \ \alpha \cdot_q x \ := \ i_q(\alpha) \cdot x,$$

(where $\cdot$ means the multiplication in $\mathbb{Q}_{(pq)}$) $\mathbb{Q}_{(pq)}$ **is not a vector space** *over $\mathbb{Q}_p$ or over $\mathbb{Q}_q$. Reely, if it would be true, then obviously we would have: $dim_{\mathbb{Q}_p} \mathbb{Q}_{(pq)} = +\infty$ and $dim_{\mathbb{Q}_q} \mathbb{Q}_{(pq)} = +\infty$, what is impossible, since it is well-known that LCA-vector spaces over local fields must be finite-dimensional (see e.g. [W, I.2 Corrolary 2]). Moreover, according to the* **Dantzing's description** *of local fields, all extensions of p-adic number fields $\mathbb{Q}_p$, must be finite extensions of such fields! (3).* **The tensor product rings** $\mathbb{Q}_p \otimes_{\mathbb{Q}} \mathbb{Q}_q$.

Let us observe that our main bi-adele (small pq-adele) ring $\mathbb{Q}_{pq} = \mathbb{Q}_p \times \mathbb{Q}_q$ is sufficiently good and "rich" for our purpose, since from the point of view of the Haar-module theory the set of its all **non-invertible** *elements : $(\mathbb{Q}_{pq}^*)^c := \mathbb{Q}_{pq} - \mathbb{Q}_{pq}^*$ is "small", i.e. its* **Haar measure** *is zero : $(H_p \times H_q)((\mathbb{Q}_{pq}^*)) = 0$.*

For each $x \in \mathbb{Q}_{pq}^*$ by $\Delta_{pq}(x)$ (or $mod_{\mathbb{Q}_{pq}}(x)$) we denoted the **Haar module** *of the* **automorphism** *$x \longrightarrow a \cdot x$ of $(\mathbb{Q}_p \times \mathbb{Q}_q)^+$. Thus*

$$\Delta_{pq}(x) \ = \ mod_{\mathbb{Q}_{pq}}(x) \ := \ \frac{(H_p \times H_q)(xX)}{(H_p \times H_q)(X)},$$

for arbitrary measurable set X in $\mathbb{Q}_{pq}$ with $0 < (H_p \times H_q)(X) < +\infty$ (for example, for X we can take any compact neighbourhood of zero).

According to the Weil's Lemma 2 (see [W, I.2 , Lemma2]) we have

$$\Delta_{pq}(x) \ = \ mod_{\mathbb{Q}_{pq}}(x) \ = \mid x_p \mid_p \mid x_q \mid_q \ , \ x = (x_p, x_q) \in \mathbb{Q}_{pq}^*.$$

The above formula suggests that we can also descibe the pq-vectors from $\mathbb{Q}_{pq}$ *in the terminology of the* **Grothendieck** *tensor products.*

Let us consider the **algebraic** *tensor product* $\mathbb{Q}_p \otimes_{\mathbb{Q}} \mathbb{Q}_q$ *and the natural map* $t_{pq} : \mathbb{Q}_p \otimes_{\mathbb{Q}} \mathbb{Q}_q \longrightarrow \mathbb{Q}_{(pq)}$ *defined by*

$$t_{pq}(x \otimes y) \; = \; xy \; , \; x \in \mathbb{Q}_p, \; y \in \mathbb{Q}_q,$$

(althought, according to the above mentioned troubles with the multiplication in $\mathbb{Q}_{(pq)}$ *it is not* **algebraic***).*

The Grothendieck π*-norm* $| \cdot |_p \otimes_\pi | \cdot |_q$ *(of* $| \cdot |_p$ *and* $| \cdot |_q$*, see e.g. [MT]) is denoted in the sequel by* $| \cdot |^\pi_{pq}$ *and is defined by*

$$| \; x \; |^\pi_{pq} := \; (| \cdot |_p \otimes | \cdot |_q)(x) \; :=$$

$$= \; inf\{\sum_i \; | \; x_i \; |_p | \; y_i \; |_q : \; x = \sum_i x_i \otimes y_i \; ; \; x_i \in \mathbb{Q}_p, y_i \in \mathbb{Q}_q\}.$$

The above π*-norm is a* **cross-norm** *(of* $| \cdot |_p$ *and* $| \cdot |_q$*), i.e.*

$$(| \cdot |_p \otimes_\pi | \cdot |_q)(x_1 \otimes x_2) \; = | \; x_1 \; |_p \cdot | \; x_2 \; |_q \; ; \; x_1 \in \mathbb{Q}_p, x_2 \in \mathbb{Q}_q.$$

Moreover

$$| \; x_1 \otimes x_2 \; |^\pi_{pq} = | \; x_1 \; |_p | \; x_2 \; |_q = mod_{\mathbb{Q}_{pq}}((x_1, x_2)) =$$

$$= \; \Delta_{pq}((x_1, x_2))$$

$x_1 \in \mathbb{Q}_p, x_2 \in \mathbb{Q}_q$ *and that norm is* **archimedean**. *By* $\mathbb{Q}_p \hat{\otimes}_\pi \mathbb{Q}_q =: \mathbb{Q}^\otimes_{pq}$ *we denote the* **completion** *of* $(\mathbb{Q}_p \otimes_{\mathbb{Q}} \mathbb{Q}_q, | \cdot |^\pi_{pq})$.

Obviously $\mathbb{Q}^\otimes_{pq}$ *is a LCA-ring. Let us denote by* H_{pq} *its* **Haar measure**.

Let $i_{pq} : \mathbb{Q}_p \times \mathbb{Q}_q \longrightarrow \mathbb{Q}^\otimes_{pq}$ *be the canonical inclusion homomorphism, i.e.* $i_{pq}(x, y) = x \otimes y$.

Since we have got **Haar measures** H_p *and* H_q *of* $\mathbb{Q}^+_p$ *and* $\mathbb{Q}^+_q$*, respectively, then we can define their* **tensor product** $H_p \otimes H_q$ *on the LCA-subring* $Im(i_{pq})$ *:*

$$i^*_{pq}(H_p \times H_q) \; =: \; H_p \otimes_\pi H_q.$$

It is easy to check that the tensor product $H_p \otimes_\pi H_q$ *of Haar measures is a* **Haar measure** *on* $Im(i_{pq})$.

Thus, since the Haar measure H_{pq} *of* $\mathbb{Q}_p \hat{\otimes}_\pi \mathbb{Q}_q$ *is unique (up to a constant), then we can assume that*

$$H_{pq} \; = \; H_p \otimes_\pi H_q,$$

i.e. the Haar measure H_{pq} *is the* π*-tensor product of the Haar measures of* $\mathbb{Q}^+_p$ *and* $\mathbb{Q}^+_q$ *(see [MT]).*

Finally, let us remark that the tensor products of gaussian measures in Banach spaces were firstly considered by **R. Carmona** *and* **S. Chevet** *In [*M_T*] were defined and considered tensor products of p-stable measures with* $0 < p < 2$*, in Banach spaces of stable*

type. The above considered tensor products of Haar measures are also tensor products of measures in the sense of the definition given in [M_T].

*(4)***The adele ring** $\mathbb{Q}_{\mathbb{A}} \subset \prod_{p\in P} \mathbb{Q}_p \times \mathbb{Q}_\infty (; = \mathbb{R})$ **of** $\mathbb{Q}$.

Ideles *were introduced by* **C. Chevalley** *in [Ch] in 1936.* **E. Artin** *and* **G. Whaples** *occured* **adeles** *in [AW], where they are called valuation vectors. The ring of adeles admits* **K. Iwasawa's characterisation** *(see [I]) in the following way : if R is a semi-simple commutative LCA-ring with a unit element, which is neither compact nor discrete, and there is a field $K \subset R$, with the same unit element, which is discrete and such that R/K is compact, then R is the* **ring of adeles** *either over an algebraic number field or over an algebraic function field with a finite fields of constants.*

Topological properties of adeles and ideles were investigated by E. Artin, K. Iwasawa, T. Tamagawa and J. Tate (see e.g.[N, Chapter VI]).

At the end of the ends, all the above considered versions of $\mathbb{Q}_{pq}$ are closely related to each other and moreover we have the following inclusion :

$$\mathbb{Q}_{pq} \subset \mathbb{Q}_p \otimes_{\mathbb{Q}} \mathbb{Q}_q \subset \mathbb{Q}_{(pq)} \subset \mathbb{Q}_{\mathbb{A}}.$$

In the sequel we denote one of the two LC rings above by R, i.e. $R = \mathbb{H}$ or $\mathbb{Q}_{pq}$. We also simply write $(\Delta, H, R, \beta, X(M,N))$ instead of $(\Delta_R, H_R, R_R, \beta_R, X_R(M,N))$. Then we have the following shocking result (a constructive mathematical construction)

Theorem 2 *(***The existence of singular solutions of** *AFE_V).*

Let V be a **real vector space** *with a* **continuous idempotent endomorphism** *$F : V \longrightarrow V$. Then* **each element** *$v_0 \in V$ is a* **±-fixed point of** *F, i.e.*

$$V \;=\; Fix(F), \tag{3.78}$$

and moreover an arbitrary $v_0 \in V$ has the following **Riesz-Bogoluboff-Kriloff Abstract Hodge Decomposition (Representation)***(AHD_{RBK} in short)*

$$(AHD_{RBK})\; v_0 \;=\; \int\int_{\mathbb{S}\times X(M,N)} [(I \pm F)(v(\Delta^2(r), v_0))] d(\beta \otimes R)(s,r). \tag{3.79}$$

Proof. Since $R = \mathbb{H}$ or $\mathcal{R} = \mathbb{Q}_{pq}$ and according to Lemma 2 , for each v_0 we have at our disposal the whole family

$$\mathcal{V}_\pm(v_0) \;:=\; \{v_\pm(l, v_0) : l \neq \pm 1\} \tag{3.80}$$

of solutions of the family of the **abstract Fox equations**

$$v_\pm(l, v_0) \;\pm\; lFv_\pm(l, v_0) \;=\; v_0. \tag{3.81}$$

We substitute $l = \Delta^2(r), r \in R^*, \Delta(r) \neq 1$, in (3.81), thus obtaining

$$\frac{v_\pm(\Delta^2(r), v_0)}{\Delta^2(r)} \;+\; F(v_\pm(\Delta^2(r), v_0)) \;=\; \frac{v_0}{\Delta^2(r)}. \tag{3.82}$$

Integrating both sides of (3.82) with respect to the Haar measure H on $X(M,N)$ and applying formula (3.67), here in the form

$$\int_{\mathcal{R}^*} f(r)\chi_{X(M,N)}(r)dH(r) \ = \ \int_{\mathcal{R}^*} f(r^{-1})\frac{\chi_{X(M,N)}(r^{-1})}{\Delta^2(r)}dH(r), \tag{3.83}$$

we obtain the equality

$$\int_{X(M,N)} \Delta^{-2}(r)v_{\pm}(\Delta^2(r),v_0)dH(r) \pm \int_{X(M,N)} \Delta^{-2}(r)F(v_{\pm}(\Delta^{-2}(r),v_0))dH(r) \ = \tag{3.84}$$

$$= v_0\int_{X_{M,N}} \Delta^{-2}(r)dH(r) \ =: \ v_0 m_{-2}(M,N),$$

where we denote the $(-2)-R-$**moment** of the Haar measure H on the compact $X(M,N)$ by $m_{-2}(M,N)$.

Let us consider the expressions

$$\int_{X(M,N)} \frac{v_{\pm}(\Delta^{\pm 2}(r),v)dH(r)}{\Delta^2(r)}. \tag{3.85}$$

Applying the **compact-R-Hilbert transform** $\mathcal{H}$ in the form of the Abstract Hodge Decomposition (AHD_R) :

$$\Delta(h^{-2}) \ = \ \int_{\mathbb{S}} \frac{dR(y)}{\Delta(h^2-y^2)} \ = \ \mathcal{H}(R)(h^2), \tag{3.86}$$

$h \in X(M,N)$, and using the **Fubini theorem** we obtain

$$\int_{X(M,N)} \frac{v_{\pm}(\Delta^{\pm 2}(r),v_0)dH(r)}{\Delta(r^2)} = \int_{X(M,N)} v_{\pm}(\Delta^{\pm 2}(r),v_0)\int_{\mathbb{S}} \frac{dR(y)}{\Delta(r^2-y^2)}dH(r) = \tag{3.87}$$

$$= \int_{\mathbb{S}} dR(y)\int_{X(M,N)} \frac{v_{\pm}(\Delta^{\pm 2}(r),v_0)dH(r)}{\Delta(r^2-y^2)}.$$

But, according to the formula (W_i), we can write the second inner integral in the iterated integral above in the form : (since $\Delta(y)=1, r/y =: r'$)

$$\int_{X(M,N)} \frac{v_{\pm}(\Delta^{\pm 2}(\frac{r}{y}),v_0)dH(r)}{\Delta^2(y)\Delta(1-(\frac{r}{y})^2} = \Delta(y^{-3})\int_{X_{M,N}} \frac{v_{\pm}(\Delta^{\pm 2}(r),v_0)dH(r)}{\Delta(1-r^2)} \ , \ y \in \mathbb{S}. \tag{3.88}$$

But $\frac{dH(r)}{\Delta(1-r^2)}$ $=: \ dHer(r) \ = \ d\beta(r)$ is the **Herbrand distribution** of the invertion $I = I_{\mathcal{R}}$ of $\mathcal{R}^*$, i.e.

$$\int_{X(M,N)} \frac{v(\Delta^2(r),v_0)dH(r))}{\Delta(1-r^2)} \ = \ \int_{X(M,N)} \frac{v(\Delta^{-2}(r),v_0)dH(r)}{\Delta(1-r^2)}, \tag{3.89}$$

since, for each integrable function ϕ the following is true

$$\int_{X(M,N)} \phi(I(r))dHer(r) \;=\; \int_{X(M,N)} \phi(r)d(I^*Her)(r) \;=$$

$$= \int_{X(M,N)} \phi(r)dHer(r).$$

Let us set:

$$v_1^{\pm} \;:=\; \int_{\mathbb{S}} \Delta(y^{-3})dR(y) \int_{X(M,N)} \frac{v(\Delta^2(r), v_0)dH(r)}{\Delta(1-r^2)}. \tag{3.90}$$

Since our "manipulations" only acted up on the parameters l and under the assumption, F is continuous and linear, then we finally obtain the RBK-integral representation above, which at the same time is the **singular solution** of (AFE_V) :

$$v_1^{\pm} \;\pm\; F(v_1^{\pm}) \;=\; m_{-2}v_0. \tag{3.91}$$

Remark 3 *(***On a shocking consequence of the construction of Th.2. The mathematics and logic***).*

Obviously, the thesis (3.78) is not true (on the ground of classical logic) for the majority of idempotent pairs (V, F). Reely, let $V = \mathbb{C}$ be considered as the 2-dimensional Banach space over $\mathbb{R}$ and let $F = c$ be the complex conjugation. Then

$$Fix(c, \mathbb{C}) \;=\; \mathbb{R} \neq \mathbb{C}.$$

The construction in Th.2 is a following step in the old and well-known philosophical problem : what is the connection between maths and (classical) logic?

As it is well-known, **Frege** *saw mathematics as only a part of logic (more exactly, according to Frege, the whole of mathematics can be reduced to logic).*

Probably the first mathematician, who questioned Frege's approach to mathematics was pre-intuitionist **Kronecker**. *He attacked well-known* **Cantor's proof** *(in "naive" set theory), of the existence of* **transcendental numbers** *$t \in T$.*

Let $\tilde{\mathbb{Q}}$ be the (algebraically closed) field of **algebraic numbers** *and assume that TnD is true :*

$$v_{Cl}(p\vee \sim p) \;=\; 1.$$

We can write TnD in quantifier form as the following true statement on $\tilde{\mathbb{Q}}$ (according to the laws of the quantifier calculus) :

$$(C)\forall(x \in \mathbb{R})(x \in \tilde{\mathbb{Q}})\vee \sim (\forall(x \in \mathbb{R})(x \in \tilde{\mathbb{Q}})) \;=$$

$$= \;\forall(x \in \mathbb{R})(x \in \tilde{\mathbb{Q}}) \vee (\exists(x \in \mathbb{R})(t \in T)).$$

Under the assumption, that the first term in the alternative (C) is true ($\tilde{\mathbb{Q}}$ is countable!), it follows that $\mathbb{R}$ should be countable, which is impossible, according to the well-known Cantor theorem.

Thus, according to the rules of classical calculus of statements and predicators, the second term of the alternative (C) *is true. Thus, transcendental numbers exist.*

But Cantor's reasoning does not give any information regarding a real number, which is transcendental. In other words, it does not provide a **construction** *of such a number.*

According to Kronecker, the non-constructive character of Cantor's proof of the existence of transcendental numbers is bad and hence its conclusion should be rejected. But (C) *is only a specification of TnD. Thus, questioning* (C) *is identical to questioning TnD. The immediate consequence of this was the rejection of classical logic and construction of intuitione logic by* **Heyting**. **Brouwer** *built constructive mathematics on this basis and showed that, in general, many constructions violate TnD. For example Brouwer's construction of the* **diagonal set of positive integers** $D\mathbb{N}$ *(the simplest* **Post system** *generated by a constructive object* $|$ *and the format* $\frac{x,x}{x}$ *(cf.[ML, Sect.7])) violates the statement*

$$(n \in D\mathbb{N}) \vee (n \notin D\mathbb{N}).$$

Similarly, in our case the statement

$$(v_0 \in Fix(F)) \vee (v_0 \notin Fix(F))$$

violates TnD (a real infinity exists but no a potential infinity?)

The construction in Th.2 is an example of such a construction. In reality, it leaveas out assumption : $v_0 \in Fix(F)$. *It seems that it is much worse.* **It gives a contradiction in mathematics**.

According to **Poincare**, *the only thing, which we must demand from an object which exists in mathematics is* **non-contradictivity**. *On the other hand,* **Godel's well known result** *states that it is not possible to prove the* **non-contradictivity of arithmetics of** $\mathbb{N}$ *(and , in fact, the majority of axiomatic systems). Moreover, the problem of the non-contradictivity of ZFC-set theory is much more complicated than for such arithmetics. Thus (according to* **Gentzen's non-finistic proof** *of the non-contradictivity of arithmetics), we can only* **believe** *that set theory is non-contradictory. But a belief is only a belief, and for example, the proof of Th.2 seems be done properly, according to classical logic, but it leads to classical mathematical contradiction.*

The only explanation of this phenomenon is the following : we use the methods of **measure theory** *strictly, which is subsequently based on set theory, in a strict manner. But according to the above discusion can this be ... (contradictory)?*

It is also very surprising, that such logical problems from the fundaments of mathematics appeared during the work on the Riemann hypothesis. Maybe this is one of the reasons that (RH) was unproven for so long and shows that (RH) is not a standard mathematical problem.

Finally, all the logical problems with (RH) mentioned above should lead and stimulate a subsequence discussion on mathemathical philosophy, very similar to the discourses after **Appel-Haken's proof** *of the* **four colour conjecture** *(proved with help of a computer*

program). Can we accept a proof of RH which is based - in its generality - on a theorem which leads to a contradiction although, if we bound the domain of objects to some "admissible" $v_0 \in Fix(V)$, then the construction is acceptable.

We now apply our theorem in the case $V = \mathcal{S}(\mathbb{R}^n)$ and $F = \mathcal{F}_n$. Let $A^+ = A_n^+(x)$ be a **generalized amplitude**, i.e. any function from $\mathcal{S}(\mathbb{R}^n)$ with $A^+(0) = 0$. Then, according to Th.2, there exists a **RH-fixed point** ω_A^+ (associated with A^+, cf. $[M_A, Th.2]$), i.e.

$$(\omega_A^+ \ - \ G)(x) \ = \ A^+(x) \ ; \ x \in \mathbb{R}^n. \tag{3.92}$$

As we remarked in $[M_A$, Remark 15] (see also (3.91)), ω_A^+ **cannot exist** if A is not a fixed point of $\mathcal{F}_n$, i.e. according to Remark 3.

In $[M_H$, Prop.2] we showed that a direct solution of the **RH-eigenvalue problem** exists. We constructed a concrete example of the **hermitian amplitude** $A = A_{h_0}^4$ being an eigenvalue of the parameetrized Fourier transform $\mathcal{F}_{h_0}$ and the **RH-eigenvector** ω_A^+ as the **fourth order hermite function** (cf.$[M_H$, (93.18)])

$$\omega_A^+(x) \ := \ H_{h_0}^4(x) \ := \ h_0^4 e^{-h_0^2 x^2}(16h_0^4x^4 - 48h_0^2x^2 \ + \ 12),$$

which satisfies the equation

$$\omega_A^+(x) \ - \ 12h_0^4 e^{-h_0^2 x^2} \ =: \ A(x) \ , \ x \in \mathbb{R}.$$

Here $h_0 = \frac{\sqrt{3}}{2}$ is an amplitude parameter. But it is very difficult (either we cannot or it is not possible) to find a direct analytic example of an **RH-amplitude**. The main difficulty is to find two fixed-points of the Fourier transform, which are both stricly decreasing for $x > 1$. In other words, the restriction : $\mathcal{F}_n(A^+) = A^+$ is very restrictive.

For the parameter p dependent Fourier transform $\mathcal{F}_p(f)(x) := \int_{\mathbb{R}} e^{2p^2ixy} f(y)dy$, we showed in $[M_H]$ that a direct solution of the (-)RH-eigenvector problem exists.

Defining the (-)RH-eigenvector ω_A^- as the **sixth order hermitian function** (see $[M_H]$ for details)

$$\omega_A^-(x) \ := \ H_{h_0}^6(x) \ = \ 16h_0^6 e^{-h_0^2x^2}(4h_0^6x^6 - 30h_0^4x^4 + 45h_0^2x^2 - 7.5),$$

we can define the amplitude A^- by the formula :

$$A^-(x) \ := \ \omega_A^-(x) \ + \ 60h_0^4 H_{h_0}^2(x).$$

In the last part of this paper the fundamental role plays the **second canonical Hermite function**

$$H_2(x) \ = \ \pi G(x)(4\pi x^2 \ - \ 1), \ \ x \in \mathbb{R}.$$

Integrating by parts twicely, we obtain that H_2 is a **minus fixed point** of $\mathcal{F}$: $\hat{H}_2(x) \ = -H_2(x)$.

Then, according to Th.2, there exists a (-)RH-fixed point ω_A^- (associated with an amplitude A^-), i.e.

$$(\omega_A^- \ + \ H_2)(x) \ = \ A^-(x) \ , \ x \in \mathbb{R}.$$

According to (3.91) ω_A^- cannot exists if A^- is not a minus fixed point of $\mathcal{F}_1$. But, if we take an amplitude A^- in such a way that the support of $(A^- \ - \ H_2)$:

$$supp(A^- \ - \ H_2) \ =: \ S_A$$

is the completion of a set with positive Lebesgue measure λ_n, i.e. $\lambda_n(S_A^c) > 0$, then, according to the **"separation of variables"** construction from $Th.2$ we obtain

$$supp(\omega_A^-) \ = \ S_A \ = \ and \, \mathcal{F}_1(\omega_A^-)(x) \ = \ \int_{S_A} cos(2\pi ixy)\omega_A^-(y)dy \ =: \ C_A(\omega_A^-)(x). \quad (3.93)$$

Since $C_A(H_2) \neq -H_2$, i.e. H_2 is not a **minus -fixed point** of C_A, then the calculation

$$C_A(A^-) = C_A(\omega_A^- + H_2) = C_A(\omega_A^-) + C_A(H_2) \neq -(\omega_A^- + H_2) = -A^-$$

shows that , in this case, the notion of RH-fixed point does not lead to a **contradiction** and can exist for an amplitude A^-, which is not the minus-fixed point of $\mathcal{F}_1$ (antinomies cannot be treated as a threat to the fundaments of maths or logic).

Remark 4 A. Wawrzy nczyk *(see [Wa, 3.8, Exercise 1d]) as well as we (see [M_H, Prop.1] and [M_P, Remark 1]) have considered the following example of the* **minus-fixed point** K_2 *of* $\mathcal{F}_1$ *: in the considerations concerning the* **quantum harmonic oscillator** *in quantum mechanics - one of the main roles plays the following* **second Hermite function**

$$K_2(x) \ := \ 2\pi e^{-\pi x^2}(2\pi x^2 \ - \ 1)$$

with the property : $\hat{K}_2(x) = -K_2(x)$.

However, according to **P. Biane**, *since* $G(x) := e^{-\pi x^2}$ *is a* **fixed point** *of the canonical Fourier transform* $\mathcal{F}_1$, *then integrating by parts twicely we obtain*

$$\int_{\mathbb{R}} e^{2\pi ixy}G''(y)dy \ = \ -2\pi ix \int_{\mathbb{R}} e^{2\pi ixy}G'dy \ =$$

$$= \ -4\pi^2x^2 \int_{\mathbb{R}} e^{2\pi ixy}G(y)dy \ = \ -(4\pi^2x^2G(x) \ - \ \pi G(x)) - \pi G(x).$$

Since

$$G''(x) \ = \ 2\pi G(x)(2\pi x^2 \ - \ 1) \ = \ K_2(x)$$

then

$$\hat{H}_2(x) := (G'')\hat{+}\pi G(x)(x) \ = \ -4\pi^2x^2G(x) + \pi G(x) = -H_2(x),$$

i.e. $H_2(x) := \pi G(x)(4\pi x^2 \ - \ 1)$ *is* **also** *the* **minus fixed point** *of* $\mathcal{F}_1$*!*

Since

$$H_2(x)(:= \omega_A^-(x)) \ - \ K_2(x)) \ = \ \pi G(x)(=: A^-).$$

In the sequel, we call the equality : $H_2 \; - \; K_2 = \pi G$ *- the* **BMW-example***.*

Since $A^- := \pi G$ **is not evidently the minus-fixed point of** $\mathcal{F}_1$ *(since it is the +fixed-point of* $\mathcal{F}_1$*), then the BMW-example : (1). confirms the* **correctness** *of our Th.2.-construction and RH-fixed point paradox of Remark 3. (2) We are not in possibility to explain that* **phenomena** *on the ground of the classical logic.!*

4 An (-)RH-fixed point proof of the generalized Riemann hypothesis

As opossed to the purely algebraic notion of the quasi-fixed point $Q_l(v)$ considered in the previous section, here the main part is played by a **purely analytic** notion of the (1-dimensional) **amplitude** A (cf.$[M_A]$ and $[M_H]$.

Definition 4.1 *We say that a function* $A : \mathbb{R}_+ \longrightarrow \mathbb{R}_+^*$ *is an* **PCID-amplitude***, if it is* **positive, continuous, integrable** *and* **(strictly) decreasing** *on* $[1, \infty)$*.*

The importance of PCID-amplitudes (amplitudes in short) follows from the fact that an **analytic Nakayama type lemma** holds for them (i.e. some very simple analytic statement, trivial in proof - but powerful consequences). This lemma establishes the sign of the action on the amplitude of the **plus-sine operator** S_+(cf.$[M_A, Lemma4]$ and $[M_H, Lemma2]$):

$$S_+(A)(a) \; := \; \int_0^\infty A(x)sin(ax)dx \; , \; a > 0. \tag{4.94}$$

Lemma 4 *For each* **amplitude** A *and* **frequency** $a \in \mathbb{R}_+^*$ *the following holds*

$$S_+(A)(a) \; > \; 0. \tag{4.95}$$

Proposition 3 *(***On the positivity of the Rhfe(Ace)-trace** Tr_-*).*

Let $n = [k : \mathbb{Q}]$ *and* $A^- = A_n^-(x), x \in \mathbb{R}^n$ *from* $\mathcal{S}(\mathbb{R}^n)$ *be a such function that for each e from the fundamental domain* $E(k)$*, the function* $t \longrightarrow exp(t)A_n^-(exp(t)e) =: A_n^e(t)$ *is a (1-dimensional)* **amplitude***. Then, for each complex number* $s = u + iv$ *with* $u \in (1/2, 1]$ *and* $v > 0$*, the following* **Casteulnovo-Serre-Weil inequality** *(CWS in short) holds*

$$(CWS) \; Tr_-(\zeta_k, A_n^x)(s) \; := \; \int_1^{+\infty} (t^u + t^{1-u})sin(vlogt)\theta_k(A_n^x)(t)dt \; > \; 0. \tag{4.96}$$

Proof. According to (2.35), (2.25) and (2.43) we have

$$Tr_-(\zeta_k, A_n)(s) = \sum_{0 \neq I \in \mathcal{I}_k} \sum_{u \in U(k)} \sum_{\xi \in R(I)} \int_1^\infty (\int_{E(k)} A_n((N(I)^{-1}t)^{1/n} C(\xi u) \cdot e) dH_r^0(e)) \tag{4.97}$$

$$(t^u + t^{1-u})sin(vlogt)dt/w(k).$$

Let us denote the vector

$$C(I)x^t \ := \ N(I)^{-1/n}[\sum_{j=1}^{n} x_j C_1(\alpha_j), ..., \sum_{j=1}^{n} x_j C_n(\alpha_j)].$$

After the substitution $t = e^r$ and changing of variables according to the n-dimensional substitution : $e' = e \cdot C(I)x^t$ we obtain that

$$Tr_-(\zeta_k, A_n) = \sum_{0 \neq I \in \mathcal{I}_k} \sum_{m \in \mathbb{Z}^n} \frac{1}{w(k)\Delta_r(C(I)x^t)} (\int_{E(k)} (\int_1^{+\infty} A_n^e(e^{r/n})(e^{r(u+1)} + e^{r(2-u)}) sin(vr)dr)dH_r^0(e).$$

Let us consider 1-dimensional **amplitudes** of the form

$$\mathcal{A}_n^e(r) := e^{ru}(1 + e^{r(1-2u)})A_n^e(e^{r/n}).$$

Since $\frac{d}{dr}(1 + e^{r(1-2u)}) < 0$, if $u \in (1/2, 1]$, then under our asumptions on the amplitude A_n the function $\mathcal{A}_n^e(r)$ is strictly decreasing. According to Lemma 4,

$$S_+(\mathcal{A}_n^e)(v) \ > \ 0. \tag{4.98}$$

Combining (4.97) with (4.98) we obtain the Proposition.

Remark 5 *The considered in Prop.3 the* **minus-trace** $Tr_-(\zeta_k, A_n^+)(s)$ *is obviously associated with a* **minus-fixed** *points* ω^- *of* $\mathcal{F}_n$*. We have seen that its* **positivity** *is an immediately consequence of the mentioned above analytic Nakayama lemma (Fresnel lemma).*

Instead of $Tr_-(\zeta_k, A_n^+)(s)$*, in the first version of the manuscript , we have considered the* **plus-trace** *(associated with* ω^+*)*

$$Tr_+(\zeta_k, A_n^+)(s) \ := \ \int_1^{\infty} (t^u - t^{1-u}) sin(vlogt)\theta_k(A_n^+)(t)dt,$$

(which obviously only differs from $Tr_-(\zeta_k, A_n^-)(s) =: Tr_-$ *by a* **sign** *in the subintegral expression).*

In opposite to the case of Tr_- *- the* **positivity** *of* $Tr_+ > 0$ *- as it was independly communicated to the author by the private communications by* **S. Albeverio**, **P. Biane** *and* **Z. Brze zniak**!*, is not an immediately consequence of the Fresnel lemma. In particular, the result :* $Tr_+ > 0$ *needs a machine of stochastic analysis and is a final effect of the existence of the so called* **Hodge measure** H_2^* *on* $\mathbb{C}^{++} := \{z \in \mathbb{C} : Re(s) > 0, Im(s) > 0\}$*, which gives the* **Laplace representation** *of the* **inverse of the Haar module of** $\mathbb{C}$ *:*

$$| \ z \ |^{-2} = \int_{\mathbb{C}^{++}} e^{z \cdot w} dH_2^*(w) \ \ , z \in \mathbb{C}^{++}.$$

The existence of H_2^* *is far non-obvious. Even worse, many peoples suggested to the author, that such the measure cannot exists!*

Fortunately, the problem has a positive solution, although it is a very technical and complicated in details result. So, we are not going to do it in this paper.

Lemma 5 *$M(H_2)(s)$ has no roots in the domain $\mathbb{C} - \{1, 2\}$ and has of order 1 poles at the points :2,1,0, -2,-4, ...*

Proof. Let us again recall that the canonical second Hermite function $H_2(x)$ has the form (see also [M_H, Remark 1]) :

$$H_2(x) \;=\; \pi G(x)(4\pi x^2 \;-\; 1).$$

It is easy to check (integrating by parts, see [M_H, Prop. 7]) that

$$M(H_2)(s) \;=\; (s-1)(s-2)M(G)(s-2) \;\;,\; for\; Re(s) > 2.$$

Since $M(H_2)(s)$ is well-defined for $Re(s) > 0$ (because $H_2 \in \mathcal{S}(\mathbb{R})$), then the above formula gives the analytic continuation of the previous right-hand side formula - defined for $Re(s) > 2$.

Making the substitution $\pi x^2 = t$ in Gamma integral we obtain

$$M(G)(s) = \int_0^\infty x^{s-2} e^{-\pi x^2} dx = \frac{\pi^{1-s}}{2\pi}\Gamma(s/2),$$

where Γ denotes the classical gamma function. Since $M(G)(s)$ **does not vanish anywhere**, then $M(H_2)$ **does not vanish** for $\mathbb{C}_2 := \mathbb{C} - \{1, 2\}$.

The final preliminary result which is very convenient when we work with (gRH_k) is the elegant **Rouche theorem**(cf.e.g. [Ma, Th. XV.18]) : let $\Omega \subset \mathbb{C}$ be a **domain** and $D \subset \Omega$ be **compact**. Let f and g be **holomorphic functions** on Ω, which satisfy the two following **Rouche's border conditions**:

$$f(z) \;\neq\; 0 \; for \; z \in \partial D \; (the\; border\; of\; D), \tag{4.99}$$

and

$$|\; g(z) \;| < |\; f(z) \;| \;\; for \; z \in \partial D. \tag{4.100}$$

Then the number of zeros $N_D(f+g)$ of the sum $f+g$ in D (weighed by their orders) is equal to the number of zeros $N_D(f)$ of f in D (**Rouche's thesis**, the adic type behaviour of N_D), i.e.

$$N_D(f+g) \;=\; N_D(f). \tag{4.101}$$

Proposition 4 *(***A Rouche choice of the amplitude A^+ and lack of roots of** *$\Gamma_r(G+A^+)$). We can choose a plus amplitude A_n^+ in such a way that : (1) the construction of the plus RH-fixed point ω_A^+ in Th.2 fulfills all the rigours of the classical logic, i.e. it does not violate TnD.*

(2) $\Gamma_r(G + A_n^+)(s) \neq 0$ for $Re(s) > 0$.

Proof. We use the Rouche theorem in the case : $\Omega = \mathbb{C}$,

$$D \;=\; D_M \;:=\; \{s \in \mathbb{C} : Re(s) \in [0, 1], Im(s) \in [-M, M]\}, \; M > 0$$

and

$$f(z) \;=\; \Gamma_r(G)(z) \;\;,\;\; g(z) \;=\; \Gamma_r(A_n^+)(z).$$

Since obviously $\Gamma_r(G)(z) \neq 0$ for $z \in \partial D_M$ and $N_{D_M}(\Gamma_r(G)) = 0$, then it suffices to show that

$$\mid \Gamma_r(A_n^+)(z) \mid^2 < \mid \Gamma_r(G)(z) \mid^2 \tag{4.102}$$

for $z \in D_M$, to conclude that $N_{D_M}(G + A_n^+) = 0$ in D_M.

The inequality (4.102) is obviously equivalent to the inequality

$$Re^2(\int_{G_r} mod_r(g)^{s-1} A_n^+(g) d^n g) \;+\; Im^2(\int_{G_r} mod_r(g)^{s-1} A_n^+(g) d^n g) < \tag{4.103}$$

$$< \; Re^2(\int_{G_r} mod_r(g)^{s-1} G_n(g) d^n g) \;+\; Im^2(\int_{G_r} mod_r(g)^{s-1} G_n(g) d^n g.$$

Let us consider the Taylor expansion of $G_n(x) = e^{-\pi||x||^2}$

$$G_n(x) \;=\; \sum_{m=0}^{\infty} \frac{(-1)^m \pi^{2m} \mid\mid x \mid\mid_n^{2m}}{m!}, \; for \; x \in \mathbb{R}^n,$$

and let us denote $g_m := \frac{\pi^{2m}}{m!}$.

Without loss of generality we can assume that A_n^+ is NCID-amplitude, i.e. is **negative** continuous integrable and such that $-A_n^+$ is strictly decreasing for $\mid\mid x \mid\mid_n \geq 1$. Reely, taking s with $Im(s) < 0$ we obtain : $Tr_+(\zeta_k, -A_n^+) > 0$. We then can define A_n^+ as follows

$$A_n^+(x) \;:=\; -G_n(x) \;\; for \;\;\; \mid\mid x \mid\mid_n \geq 1, \tag{4.104}$$

and

$$A_n^+(x) \;:=\; -\sum_{m=2}^{\infty} (-1)^m g_{m-2} \mid\mid x \mid\mid_n^{2m} \;\;\; if \;\;\; \mid\mid x \mid\mid_n \in [0,1]. \tag{4.105}$$

Since $\sum_{m=2}^{\infty} (-1)^m g_{m-2} \;=\; \sum_{m=0}^{\infty} g_m$, then A_n^+ is **continuous** and hence - NCID-amplitude.

Moreover from the definition we get that the support of $(G \;+\; A^+)$ is the **unit ball** B_n of $\mathbb{R}^n$.

Thus, to obtain (4.103) it suffices to show that for $Re(s) \geq 0$ holds

$$\mid Re(\int_{G_r \cap B_n} mod_r(g)^s A_n^+(g) dH_r(g)) \mid < \mid Re(\int_{G_r \cap B_n} mod_r^s G_n(g) dH_r(g)) \mid, \tag{4.106}$$

and for $Im(s) \leq 0$ holds

$$\mid Im(\int_{G_r \cap B_n} mod_r(g)^s A_n^+(g) dH_r(g)) \mid < \mid Im(\int_{B_n \cap G_r} mod_r(g)^s G_n(g) dH_r(g)) \mid, \tag{4.107}$$

since, according to the definition of A_n^+ we have

$$\int_{B_n^c \cap G_r} mod_r(g)^s A_n^+(g) dH_r(g) = -\int_{B_n^c \cap G_r} mod_r^s(g) G_n(g) dH_r(g) \tag{4.108}$$

and - let us recall (see (2.16) and (2.17)) -

$$\Gamma_r(\|\cdot\|_n^{2m}\,\chi_{B_n})(s) = \int_{G_r\cap B_n} mod_r(g)^s \parallel g \parallel_n^{2m} dH_r(g) \;= \tag{4.109}$$

$$\int_{G_r\cap B_n} mod_r(g)^s \parallel g \parallel_n^{2m} \frac{d^n g}{mod_r(g)} = \int\int_{(\mathbb{R}_+^* \times G_r^0)\cap B_n} mod_r^s(t^{1/n}c)t^{2m} \parallel c \parallel_n^{2m} \frac{d^n c dt}{mod_r(c)t} =$$

$$=: \; \frac{c_{2m}}{s+2m},$$

since

$$log(mod_r(g)) = \sum_{i=1}^{r_1} log \mid x_i \mid \; + \; \sum_{j=1}^{r_2} log \mid z_j \mid^2 \leq C(n) \parallel g \parallel_n^2 .$$

It is obvious that for $Re(s) \geq 0$ we have

$$\sum_{m=2}^{\infty} \frac{(-1)^m c_m g_m (Re(s)+2m)}{\mid s+2m \mid^2} < \sum_{m=0}^{\infty} \frac{(-1)^m c_m g_m (Re(s)+2m)}{\mid s+2m \mid^2},$$

since $1 \; - \; \frac{\pi(x+2)}{|x+2|^2} > 0$, according to the fact that the quadratic polynomial $x^2 + (4-\pi)x + (4-\pi) > 0$ for all $x > 0$. Analogously, for $Im(s) \leq 0$ we have

$$-Im(s) \sum_{m=2}^{\infty} \frac{(-1)^m c_m g_m}{\mid s+2m \mid^2} < -Im(s) \sum_{m=0}^{\infty} \frac{(-1)^m c_m g_m}{\mid s+2m \mid^2}.$$

Thus, according to the definition of $A_n^+(G)$, from those strict inequalities above, we claim (deduce) that the pair $(\Gamma_r(G_n), \Gamma_r(A_n^+))$ satisfies the **strong** Rouche boundary conditions (4.99) and (4.100) on every compact set $D_M, M > 0$ (and not only on ∂D_M):

$$\mid \Gamma_r(A_n^+)(s) \mid < \mid \Gamma_r(G_n)(s) \mid \,, \; s \in D_M.$$

Converging with M to the infinity we finally obtain

$$N_{D_\infty}(G_n \; + \; A_n^+) \;=\; N_{D_\infty}(G_n) \;=\; 0.$$

Proposition 5 *(***A non-contradictory choice of the amplitude** A^- **and deleting of the problem of vanishing of** $M(A^- - H_2)$*).*

We can choose an amplitude A_n^- *in such a way that :*

(1) the construction of the (-)RH-fixed point $\omega_{A_n}^-$ *in Th.2 fulfills all the rigours of classical logic, i.e. it does not violate TnD.*

(2) Even when $\Gamma_r(H_2 - A^-)(s)$ *has* **zeros** *in* $Re(s) > 0$ *then still holds the (Face*$_-$*):*

$$\Gamma_r(\omega_A^-)(s)\zeta_k(s) \;=\; \frac{\lambda_k}{s(s-1)} + \tag{4.110}$$

$$+ \int_1^\infty \int_E \theta_E(\omega_A^-)(ct^{1/n})(t^s \; + \; t^{1-s})dH_r^0(c)\frac{dt}{t}(=: \int_1^\infty \Theta_k(\omega_A^-)(t)(t^{s-1} + t^{-s})dt).$$

Proof. Let us consider the McLaurin expansion of H_2^n

$$H_2^n(x) \;=\; -\pi \;-\; \sum_{m=1}^{\infty} \frac{(-1)^m(-\pi)^{m+1}(4m+1)\,||\,x\,||_n^{2m}}{m!},$$

and let us denote $h_m := \frac{\pi^{m+1}(4m+1)}{m!}$.

For a convenience of the reader, we give here all needed in the sequel facts concerning the **graph** of H_2 (it can be easy obtained by using the elementary differential calculus). Thus :

$$H_2(0) = -\pi,\; H_2(\frac{1}{2\sqrt{\pi}}) = 0,\; H_2(1) = \pi e^{-\pi}(4\pi - 1) > 0. \tag{4.111}$$

Moreover, the function $H_2(x)$ is **positive** for $x \geq 1/2\sqrt{\pi}$ and **strictly decreasing** for $x \geq \sqrt{\frac{5}{2}}$. Finally, the sequence $\{h_m\}$ is strictly decreasing for $m \geq 4$ (see also [AM, Lemma 2]).

Looking at the graph of H_2 on $\mathbb{R}_+$, we see that we can find such $x_2 > \sqrt{5/2} > 1 > x_1 > 1/2\sqrt{\pi}$ (since $H_2(x_2) \longrightarrow +\infty$ if $x_2 \longrightarrow \infty$), that the defined below function A_n^- is an **PCID-amplitude** :

$$A_n^-(x) \;:=\; H_2^n(x) \;\; if \quad ||\,x\,||_n \geq x_2,$$

and

$$A_n^-(x) \;:=\; L(x) \; if \;\; ||\,x\,||_n \leq x_2,$$

where by L we denoted the line which connects the points $(x_2, H_2(x_2))$ and $(x_1, H_2(x_1))$ with $H_2(x_1) > H_2(x_1)$. Moreover $(H_2 \;-\; A_n^-)(x) = 0$ for $||\,x\,||_n > x_2$.

The construction of an amplitude - let us say A_n^{--} - with the property that $\Gamma_r(H_2 - A_n^{--})(s) \neq 0$ if $Re(s) > 0, Im(s) > 0$, i.e. such A_n^{--} that we could apply to it the Rouche theorem is much more technically complicated (although possible). Therefore we are not going to do it in this paper because we can overcome that problem as follows : let us observe that Th.1 gives in fact a stronger result, i.e. it holds **without any assumption** on the vanishing of $\Gamma_r(\omega_A^-)$. Reely, beside the fact that we have not any exact information on the zero-dimensional manifold $\Gamma_r(\omega_A^-)(\mathbb{C}) := \{s \in \mathbb{C} : \Gamma_r(A_n^-)(s) = 0\}$, the meromorphic functions : $\Gamma_r(A_n^-)(s)\zeta_k(s)$ and $\int_1^{\infty}(t^{s-1} - t^{-s})\Theta_k(\omega_A^-)(t)dt$ are **well-defined** for $Re(s) > 0$ and - according to (Face) - **coincides** for $Re(s) > 1$. Hence, according to the uniqueness of the continuation of the analytic functions in regions - they must be equal in $Re(s) > 0$.

Theorem 3 *(***Existence of** *$gRhfe_k^-$).*

A pair of two $\mathbf{\Gamma\theta}$**sinlog** $-$ **factors**(F_{id}, F_c) *indexed by the Galois group $Gal(\mathbb{C}/\mathbb{R}) = \{id, c\}$ and another pair (f_1, f_2) of* θ**sinlog-factors** *satisfying*

$$f_1(s) \;+\; f_2(s) \;\neq\; 0 \; for \; Re(s) \in (1/2, 1] \tag{4.112}$$

exist, such that the following $gRhfe_k^-$ (with rational term I and the action of $Gal(\mathbb{C}/\mathbb{R})$) holds

$$Im(\sum_{g \in Gal(\mathbb{C}/\mathbb{R})} (F_g\zeta_k)(g(s)) \;=\; \frac{\lambda_k(f_1(s) + f_2(s))}{|\; s(s-1)\;|} I(s). \tag{4.113}$$

Proof. (I). **The derivation of** $gRhfe_k^-$.

Let $a_2 > a_1 > 0$ be arbitrary **artificially chosen** ζ_k**-Cramer initial condition** and let $s = u + iv = Re(s) + iIm(s)$ be fixed. We consider a simple **non-homogeneous system** of two linear equations in two variables p_1 and p_2 of the form :

$$p_1 v(u-1) \ + \ p_2 vu \ = \ a_1 \ - \ a_2 \tag{4.114}$$

$$p_1 vu \ + \ p_2 v(u-1) \ = \ a_2 \ - \ a_1.$$

This system is a **Cramer system**, iff s does not belong to the algebraic $\mathbb{R}$-variety $I(\mathbb{C})$. The main determinant of (4.104) is $I(s)$ and its solution is given by the formulas

$$p_1 \ = \ p_1(Im(s)) \ = \ \frac{(a_2 \ - \ a_1)}{v} \ > \ 0 \tag{4.115}$$

and

$$p_2 \ = \ p_2(Im(s)) \ = \ \frac{(a_1 \ - \ a_2)}{v} \ = -p_1 < \ 0. \tag{4.116}$$

Let A^- be an **amplitude** chosen according to the Proposition 5. Then according to Proposition 5, there exists a **(-)RH-fixed point** ω_A^- of $\mathcal{F}_n$, i.e.

$$\omega_A^- \ + \ H_2 \ = \ A^-. \tag{4.117}$$

In the sequel we simply write $\omega_1 = \omega_A^-$. We denote the standard n-dimensional second Hermite (-)fixed point of $\mathcal{F}_n$ by $\omega_2 = H_2 = H_2^n$.

We set (cf.(2.48))

$$J_i(s) \ := \ \int_1^{+\infty} (t^{u-1} \ + \ t^{-u}) sin(vlogt) \Theta_k(\omega_i)(t) dt \ , \ i = 1, 2. \tag{4.118}$$

The integrals J_i above are **quasi-invariant** under the substitutions : $t = x^r, r > 0$, i.e. the substitution $t = x^{p_1 v}, v > 0$ gives

$$J_1(s) = p_1 v \int_1^{\infty} (x^{p_1 v(u-1)} + x^{-p_1 vu}) sin(p_1 v^2 logx) \Theta_k(\omega_1)(x^{p_1 v}) x^{(p_1 v - 1)} dx =: J_1^r(s) \tag{4.119}$$

In the same way, the substitution $t = x^{-p_2 v}, v > 0$ gives

$$J_2(s) = -p_2 v \int_1^{\infty} (x^{-p_2 v(u-1)} + x^{p_2 vu}) sin(-p_2 v^2 logx) \Theta_k(\omega_2)(x^{-p_2 v}) x^{-(p_2 v + 1)} dx =: J_2^r(s). \tag{4.120}$$

Thus, the equalities $J_i(s) = J_i^r(s), i = 1, 2$ hold on the domain $\{s \in \mathbb{C} : Im(s) \geq 0\}$. But obviously the integrals are imaginary parts of the **analytic** function $\Gamma_{(r_1, r_2)}(\omega_i)\zeta_k - \lambda_k/W$ defined on $\mathbb{C} - \{0, 1\}$. Hence, they must be equal everywhere. In particular, the second equality is **invariant** to the operation of **complex conjugation** c, i.e.

$$J_2(c(s)) = p_2 v \int_1^{\infty} (x^{p_2 v(u-1)} + x^{p_2 vu}) sin(-p_2 v^2 logx) \Theta_k(\omega_2)(x^{p_2 v}) x^{p_2 v - 1} dx = J_2^r(c(s)). \tag{4.121}$$

Since $\omega_i \in \mathcal{S}(\mathbb{R}^n)$, for each $q > 1$ we have

$$max_{x\geq 1} \mid x^q \Theta_k(\omega_i)(x^{p_i v}) \mid < \infty.$$

According to the **elementary mean value theorem**, there exists such an $x_i = x_i(s, a_1, a_2) \in [1, \infty)$ and $q = q(a_1, a_2, u) > 1$ that

$$J_i(c_i(s)) = p_i v sin((-1)^{i+1} p_i v^2 log x_i) x_i^q \Theta_k(\omega_i)(x_i^{p_i v}) \int_1^\infty (x^{p_i v(u-1)-a_i} + x^{-p_i v u - a_i}) x^{-q} dx \tag{4.122}$$

$$=: \; f_i(s) \int_i (s),$$

where $c_1 = id$ and $c_2 = c$.

The number q is obviously chossen in such a way that the integrals $\int_i(s)$ are convergent.

Using the (Face) (cf.(2.36)) and the nation from (4.122) we obtain

$$Im((\Gamma_{(r_1,r_2)}(\omega_i)\zeta_k)(c_i(s)) = \frac{\lambda_k I(c_i(s))}{\mid s(s-1) \mid^2} + f_i(s) \int_i (s), \tag{4.123}$$

or equivalently

$$Im((\Gamma_{(r_1,r_2)}(\omega_1) f_2 \zeta_k))(s) = \frac{(f_2 I)(s)}{\mid s(s-1) \mid^2} \; + \; (f_1 f_2)(Im(s)) \int_1 (s), \tag{4.124}$$

together with

$$Im(\Gamma_{(r_1,r_2)}(\omega_2) f_1 \zeta_k)(c(s)) = \frac{-(f_1 I)(s)}{\mid s(s-1) \mid^2} \; + \; (f_1 f_2)(Im(s)) \int_2 (s). \tag{4.125}$$

By defining the $\Gamma\theta$ **sinlog-factors** as

$$F_{id}(s) := (\Gamma_{(r_1,r_2)}(\omega_1 f_2))(s) \; and \; F_c(s) := (\Gamma_{(r_1,r_2)}(\omega_2 f_1))(s), \tag{4.126}$$

and substrating (4.125) from (4.124), according to the choice of the pair (p_1, p_2) in (4.114) (which is the solution of the Cramer system) we finally obtain $(gRhfe_k^-)$.

(II).**Positivity of** $Tr(\zeta_k, A)$ (It is a very subtle "game" of signs - on the bourder of subtlety) .

According to the construction of ω^A, we have

$$A \; = \; \omega_1 \; + \; \omega_2. \tag{4.127}$$

By Proposition 3 on the positivity of the trace, we have

$$0 < Tr_-(\zeta_k, A)(s) \; = \; (J_1 + J_2)(s) \; = \; J_1(s) + J_2(m(s)) \; = \tag{4.128}$$

$$J_1(s) \; + \; J_2(c(a(s))),$$

where - for a moment - we denoted the affinic antyconjugation as $a(s) := (1-u)+iv$, and

$$J_2(s) \ = \ Im(\int_1^{\infty}(t^{s-1}-t^{-s}))\Theta_k(t)dt = -J_2(m(s)). \tag{4.129}$$

Moreover, on the basis of the notation in (4.122) we have

$$f_2(s) \ = \ \frac{J_2(c(s))}{\int_2(s)},$$

and therefore

$$f_2(a(s)) \ = \ -f_2(s) \ and \ \int_2(a(s)) \ = \ \int_2(s). \tag{4.130}$$

Since the pair (p_1,p_2) is the solution of the Cramer system (4.104), we obtain

$$-\int_2(s) \ = \ \int_1(s). \tag{4.131}$$

Hence, combining (4.128), (4.130) and (4.131) we finally obtain

$$0 < Tr_-(\zeta_k, A)(s) = f_1(s)\int_1(s) \ + \ f_2(a(s))\int_2(a(s)) \ = \tag{4.132}$$

$$= \ \int_1(s)(f_1(s) \ + \ f_2(s)),$$

i.e.

$$f_1(s) \ + \ f_2(s) \ \neq \ 0 \ for \ Re(s) \in (1/2, 1]$$

which proves Theorem 3.

Remark 6 *It is a very exciting fact that to prove (gRH_k) we need only* **two** *functional equations for $\zeta_k(s)$!, whereas - among number theory specialists - we have met with the quite opposite opinion - that even infinitely many f.e. for $\zeta_{\mathbb{Q}}(s)$ are* **not sufficient** *to proof (RH)! (e.g. H. Iwaniec).*

Obviously $(gRhfe_k^-)$ immediately implies the **generalized Riemann Hypothesis**. Assume that there exists a zero s_0 of ζ_k in the set $\{s \in \mathbb{C} : Re(s \in (1/2, 1]\}$. Then

$$\sum_{g \in Gal(\mathbb{C}/\mathbb{R})} (F_g\zeta_k)(g(s_0)) \ = \ 0,$$

since, according to HRace, the zeros of zeta lie symmetrically with respect to the lines : $Im(s) = 0$ and $Re(s) = 1/2$. But, on the other hand, we have

$$\frac{(f_1 \ + \ f_2)(s_0)}{\mid s_0(s_0-1) \mid^2} I(s_0) \ \neq \ 0,$$

which is impossible according to $(gRhfe_k^-)$.

Remark 7 *The CWS-inequality*

$$Tr^k_{Gal(\mathbb{C}/\mathbb{R})}(s) \ = \ Tr^k_G(s) \ := \ \frac{\lambda_k(f_1+f_2)(s)}{\mid s(s-1) \mid^2} \ > \ 0,$$

is **exceptional***(fundamental) to the proof of* (gRH_k). *That is very surprising that similar kinds of positivity conditions (explored also in [*M_A*],* $[M_H]$ *and [AM]) are strictly connected with (RH) :*

In $[M_{CG}]$*, based on* $[M_L]$ *we showed that the positivity of the Cauchy-Gaussian trace* Tr_{CG} *implies the Riemann hypothesis.*

In [B] de Branges showed that the positivity of his trace Tr_B *would imply the Riemann hypothesis (also in the case of some L-functions).*

Below we briefly remind the reader that the positivity of the **Weil trace** Tr_W *leads to the Riemann hypothesis.*

As it is well-known (cf.e.g.[L, XVII.3]), A. Weil formulated an equivalent form of the Riemann hypothesis (the **Weil Formula** *(WF in short)) in terms of the* **positivity** *of his functional: let* $\mathcal{SB}(\mathbb{R})$ *be the* **restricted Barner-Schwartz space** *of all functions of the form*

$$F(x) \ = \ P(x)e^{-Kx^2}$$

with some real constant $K > 0$ *and some polynomial* P *(cf.[L,XVII.3]). Then* $\mathcal{SB}(\mathbb{R})$ *is self-dual, and functions from this space satisfy the* **three Barner conditions** *(cf.[L]) : finitness of variation, Dirichlet normalization and asymptotic symmetry at zero.*

For each $F \in \mathcal{SB}(\mathbb{R})$ *is well-defined its* **conjugation**

$$F^*(x) \ := \ F(-x),$$

and F *is of* **positive type** *if* F *is equal to its* **Rosatti convolution**

$$F \ = \ F_0 * F_0^*,$$

for some $F_0 \in \mathcal{SB}(\mathbb{R})$. *(So we see that* $\mathcal{SB}(\mathbb{R})$ *is also closed under the convolution* $*$*). For* $s = \sigma + it$*, we can consider the* **two-sided Laplace-Fourier transform**

$$\hat{F}(s) \ := \ \int_{\mathbb{R}} F(x)e^{(1/2-\sigma)x}e^{itx}dx,$$

and the **Weil functional** W *defined as*

$$W_k(\Phi) \ := \sum_{\rho,\zeta_k(\rho)=0,Im(\rho)\neq 0} \Phi(\rho).$$

Then the Riemann hypothesis is equivalent to the positivity of Weil's trace

$$Tr_W(F_0) \ := \ W_k(F_0 * F_0^*) \ \ \geq \ 0, \tag{4.133}$$

for all $F_0 \in \mathcal{SB}(\mathbb{R})$.

Weil's condition *is much more general.*

Let k be a number field, χ a **Hecke character**, *$\{_\chi$ the* **conductor**, *$\mathcal{D}$ the* **local different** *and $d_\chi = N(\mathcal{D}f_\chi)$.*

Let us consider the L_k^*-function

$$L_k^*(s;\chi) \;:=\; [(2\pi)^{-n(k)}2^{r_1}d_\chi]^{s/2} \prod_{v\in S_\infty(k)} \Gamma(s_v/2)L(s;\chi), \tag{4.134}$$

where $L(s;\chi)$ is the **Hecke L-function** *associated with χ, i.e. the usual product over unramified prime ideals for χ and $s_v := N_v(s+i\phi_v)+ \mid m_v \mid$ (cf.[L]).* **Weil's functional** *W in this case is obviously the sum*

$$W_L(F) \;=\; \sum_{L(\rho,\chi)=0} F(\rho) \;\;,\;\; F \in \mathcal{SB}(\mathbb{R}).$$

In short, the **generalized Riemann hypothesis** *for $L_k^*(\cdot;\chi)$, $gRH_k(\chi)$, states that $Re(\rho) = \frac{1}{2}$ for all zeros ρ of $L_k(\cdot;\chi)$ in the critical strip. Well-known* **Weil's theorem***(cf.[L,Th.3.3]) asserts that $gRH_k(\chi)$ is equivalent to the property that*

$$(WC)\forall(F_0 \in \mathcal{SB}(\mathbb{R}))(W_L(F_0 * F_0) \geq 0). \tag{4.135}$$

In particular, we have thus proved Weil's theorem for the Dedekind zetas.

Theorem 4 *For all F_0 in the restricted Schwartz space $\mathcal{SB}(\mathbb{R})$ the following holds*

$$W_{\zeta_k}(F_0 * F_0^*) \;\geq\; 0.$$

5 The generalized Riemann hypothesis for all Dirichlet L-functions

A first generalization of the Riemann zeta function comes from **Dirichlet**[Di], who for a character χ of $(\mathbb{Z}/m\mathbb{Z})^*$, that is, a homomorphism from $(\mathbb{Z}/m\mathbb{Z})^*$ to $\mathbb{C}^*$, considered the series

$$L(s,\chi) \;:=\; \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \tag{5.136}$$

where $\chi(n) := \chi([n])$ for $(n,m) = 1$ and $\chi(n) = 0$ for $(n,m) \neq 1$. He used these L-series to prove his theorem on primes in arithmetic progressions, in which of principal importance is the fact that the value of $L(s,\chi)$ is nonzero at the point $s = 1$.

Let m be a natural number amd ζ_m a primitive mth root of unity, that is, a complex number with $\zeta_m^m = 1$ and $\zeta_m^i \neq 1$ for $1 \leq i \leq m$. In this section we consider extensions k that arise from $\mathbb{Q}$ through the adjunction of roots of unity. The field $k = \mathbb{Q}(\zeta_m)$ is called the **mth cyclotomic field**, since as points in the complex plane they divide the circle into equal arcs (see [K, Sect. 6.4]).

Since the development by Kummer of the theory of cyclotomic fields (see e.g. [K]) one proves $L(1, \chi) \neq 0$ for characters χ different from the **trivial character** χ_0 ($L(s, \chi_0)$ has a simple pole at $s = 1$) most naturally with the help of the following result (see [K, Sect.8.2, Th.8.2.1.]) :

for any integer $m \in \mathbb{N}$

$$\zeta_{\mathbb{Q}(\zeta_m)}(s) \;=\; \prod_{p|m}(1 \;-\; \frac{1}{N(p)^s})^{-1} \prod_{\chi} L(s, \chi), \tag{5.137}$$

where the right-hand product runs over all characters of $(\mathbb{Z}/m\mathbb{Z})^*$.

Theorem 5 *(gRH_m **for Dirichlet L-functions**) Let m be any positive integer and $\chi_m : \mathbb{F}_m^* = (\mathbb{Z}/m\mathbb{Z})^* \longrightarrow \mathbb{C}$ any character of the multiplicative group of the finite ring $\mathbb{F}_m$. Let also χ_m be corresponding* **Dirichlet character**. *Then the following implication is true :*

$$(gRH_m)\; If\; L(s, \chi_m) = 0 \; with \; Im(s) \neq 0 \; then \; Re(s) = 1/2. \tag{5.138}$$

In particular, the **Weil trace** $Tr_{W,m}(F_0) := \sum_{\rho, L(\rho,\chi_m)=0, Im(\rho)\neq 0}(F_0(\rho) * F_0(\rho))$ *associated with the L-function $L(s, \chi_m)$ is* **positive**, *i.e.*

$$Tr_{W,m}(F_0) \;\geq\; 0, \tag{5.139}$$

for all F_0 from the Barner-Schwartz space $\mathcal{SB}(\mathbb{R})$.

Proof. Assume (a contrary) that there is a **zero** s_0 of $L(s, \chi_m)$ in the domain : $Re(s) \in (0, 1) - \{1/2\}, Im(s) \neq 0$ of $\mathbb{C}$. Then, according to the "spliting formula" (5.137) we obtain that

$$\zeta_{\mathbb{Q}(\zeta_m)}(s_0) \;=\; 0,$$

what obviously is not possible according to gRH_k.

Thus, the generalized Riemann hypothesis for Dirichlet L-functions - according to (5.137) - is directly and immediately reduced (or is the consequence) of the generalized Riemann hypothesis for Dedekind zetas - proved in the previous Section.

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e-mail: madrecki@im.pwr.wroc.pl